Journal of
Mathematics
and Applications

JMA No 46, pp 115-148 (2023)

Bayer Noise Symmetric Functions and Some Combinatorial Algebraic Structures

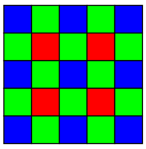
Adnan Hashim Abdulwahid and Elgaddafi Elamami

Abstract: Symmetric functions play a crucial role in classifying representations of symmetric groups, and they are largely involved with combinatorial algebras and graph theory. Bayer filter technique is largely applied in most of the professional digital cameras due to the fact that it is a low-cost, and it allows photosensors not only to capture the intensity of light, but also to record the wavelength of light as well. Using Bayer Pattern, we introduce the Bayer Noise symmetric functions and the Bayer Noise Schur functions, and we study some combinatorial structures on the Bayer Noise modules. We study the connection between Bayer Noise symmetric functions and other bases for the algebra of symmetric functions, and we explicitly calculate special cases over a fixed commutative ring $\mathbf{k}$. We also study the compatibility of such algebraic and coalgebraic structures.

AMS Subject Classification: 05E05, 05E40, 05E18, 05E15, 16T15.
Keywords and Phrases: Symmetric functions; Schur functions; Algebra; Coalgebra; Noise.

1. Introduction

A Bayer filter mosaic is a color filter array by which RGB color filters are arranged on a square grid of photosensors. This approach is very common and applied in most single-chip digital image sensors and extensively in professional equipment. Half (50%) of the filter elements are green and the rest are composed of blue and red (25% red and 25% blue). This gives an approximation for human photopic vision where the M and L cones amalgamate to produce a bias in the green spectral region [1, p. 124].



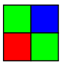
Bayer Filter Mosaic (in terms of colors)

B	G	B	G	B
G	R	G	R	G
B	G	B	G	B
G	R	G	R	G
B	G	B	G	B

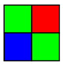
Bayer Filter Mosaic (in terms of letters).

Basically, there are four patterns of this filter: GBRG, GRBG, BGGR and RGGB. A Bayer pattern array can be shown in the following figure.

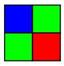
There are basically four patterns of this filter: GBRG, GRBG, BGGR and RGGB.



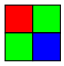
GBRG Pattern



GRBG Pattern



BGGR Pattern



RGGB Pattern

Every BGGR-Bayer Young diagram of shape λ corresponds to a unique symmetric monomial function whose degree equals to the number of its pixels. This monomial function (which we call the Bayer Noise monomial function) can be seen as splitting an image into three parts GB-part, G-part and R-part. The GB-part can be thought of as a full-size (free color (G, B)) image (the original image) while the other parts can be seen as full-sizes (free color G) and (free color R) images respectively (see Figure 5: *Block diagram of the proposed restoration technique* in [6]). Such monomial functions allow us to define and study some interesting modules over a fixed commutative ring $\mathbf{k}$. More importantly, we study some combinatorial algebraic and coalgebraic structures on such modules. The order and color of the cells in the Bayer filter mosaic play a crucial role in defining such algebraic and coalgebraic structures.

This paper is basically an application of combinatorial algebra in image processing. To see the connection more clearly, we refer the reader to [6]. The paper is organized as follows. In section 2, we recall some basic concepts of symmetric functions. In section 3, Bayer Young diagrams and Bayer Noise monomials have been introduced. In section 4, we study some algebraic structures on Bayer Noise modules while section 5 is devoted for studying some coalgebraic structures on such modules. In section 6, we introduce Bayer Noise Schur functions, and we prove that the set of all Bayer Noise Schur functions forms another basis for the Bayer Noise module Γ.

2. Preliminaries

Throughout this paper, $\mathbf{k}$ is a commutative ring, and all unadorned tensor products are over $\mathbf{k}$. Following [2], we recall some basic concepts of symmetric functions. For the basic notions of symmetric functions, the reader is referred to [2], [3], [8], [5], [11], [10], [4] or [9]. Given an infinite variable set $\mathbf{x} = (x_1, x_2, \ldots)$, a monomial

$\mathbf{x}^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots$ is indexed by a sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ in $\mathbb{N}^\infty$ having finite support; such sequences α are called *weak compositions*. The nonzero entries of the sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ are called the *parts* of the weak composition α.

The sum $\alpha_1+\alpha_2+\alpha_3+\cdots$ of all entries of a weak composition $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$ (or, equivalently, the sum of all parts of α) is called the *size* of α and denoted by $|\alpha|$.

Consider the $\mathbf{k}$-algebra $\mathbf{k}[[\mathbf{x}]] := \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ of all formal power series in the indeterminates $x_1, x_2, x_3, \ldots$ over $\mathbf{k}$; these series are infinite $\mathbf{k}$-linear combinations $\sum_\alpha c_\alpha \mathbf{x}^\alpha$ (with c_α in $\mathbf{k}$) of the monomials $\mathbf{x}^\alpha$ where α ranges over all weak compositions. The product of two such formal power series is well-defined by the usual multiplication rule.

The *degree* of a monomial $\mathbf{x}^\alpha$ is defined to be the number $\deg(\mathbf{x}^\alpha) := \sum_i \alpha_i \in \mathbb{N}$. Given a number $d \in \mathbb{N}$, we say that a formal power series $f(\mathbf{x}) = \sum_\alpha c_\alpha \mathbf{x}^\alpha \in \mathbf{k}[[\mathbf{x}]]$ (with c_α in $\mathbf{k}$) is *homogeneous of degree* d if every weak composition α satisfying $\deg(\mathbf{x}^\alpha) \neq d$ must satisfy $c_\alpha = 0$. In other words, a formal power series is homogeneous of degree d if it is an infinite $\mathbf{k}$-linear combination of monomials of degree d. Every formal power series $f \in \mathbf{k}[[\mathbf{x}]]$ can be uniquely represented as an infinite sum $f_0 + f_1 + f_2 + \cdots$, where each f_d is homogeneous of degree d; in this case, we refer to each f_d as the *d-th homogeneous component of f*. Note that this does not make $\mathbf{k}[[\mathbf{x}]]$ into a graded $\mathbf{k}$-module, since these sums $f_0 + f_1 + f_2 + \cdots$ can have infinitely many nonzero addends. Nevertheless, if f and g are homogeneous power series of degrees d and e, then fg is homogeneous of degree $d + e$.

A formal power series $f(\mathbf{x}) = \sum_\alpha c_\alpha \mathbf{x}^\alpha \in \mathbf{k}[[\mathbf{x}]]$ (with c_α in $\mathbf{k}$) is said to be *of bounded degree* if there exists some bound $d = d(f) \in \mathbb{N}$ such that every weak composition $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$ satisfying $\deg(\mathbf{x}^\alpha) > d$ must satisfy $c_\alpha = 0$. Equivalently, a formal power series $f \in \mathbf{k}[[\mathbf{x}]]$ is of bounded degree if all but finitely many of its homogeneous components are zero. (For example, $x_1^2+x_2^2+x_3^2+\cdots$ and $1+x_1+x_2+x_3+\cdots$ are of bounded degree, while $x_1 + x_1x_2 + x_1x_2x_3 + \cdots$ and $1 + x_1 + x_1^2 + x_1^3 + \cdots$ are not.) It is easy to see that the sum and the product of two power series of bounded degree also have bounded degree. Thus, the formal power series of bounded degree form a $\mathbf{k}$-subalgebra of $\mathbf{k}[[\mathbf{x}]]$, which we call $R(\mathbf{x})$. This subalgebra $R(\mathbf{x})$ is graded (by degree). The symmetric group $\mathfrak{S}_n$ permuting the first n variables $x_1, \ldots, x_n$ acts as a group of automorphisms on $R(\mathbf{x})$, as does the union $\mathfrak{S}_{(\infty)} = \bigcup_{n\geq 0} \mathfrak{S}_n$ of the infinite ascending chain $\mathfrak{S}_0 \subset \mathfrak{S}_1 \subset \mathfrak{S}_2 \subset \cdots$ of symmetric groups. This group $\mathfrak{S}_{(\infty)}$ can also be described as the group of all permutations of the set $\{1, 2, 3, \ldots\}$ which leave all but finitely many elements invariant. It is known as the *finitary symmetric group* on $\{1, 2, 3, \ldots\}$. The group $\mathfrak{S}_{(\infty)}$ also acts on the set of all weak compositions by permuting their entries:

$$\sigma(\alpha_1, \alpha_2, \alpha_3, \ldots) = \left(\alpha_{\sigma^{-1}(1)}, \alpha_{\sigma^{-1}(2)}, \alpha_{\sigma^{-1}(3)}, \ldots\right)$$

for any weak composition $(\alpha_1, \alpha_2, \alpha_3, \ldots)$ and any $\sigma \in \mathfrak{S}_{(\infty)}$. These two actions are connected by the equality $\sigma(\mathbf{x}^\alpha) = \mathbf{x}^{\sigma\alpha}$ for any weak composition α and any $\sigma \in \mathfrak{S}_{(\infty)}$. The *ring of symmetric functions in* $\mathbf{x}$ *with coefficients in* $\mathbf{k}$, denoted $\Lambda = \Lambda(\mathbf{k}) = \Lambda(\mathbf{x}) = \Lambda(\mathbf{k})(\mathbf{x})$, is the $\mathfrak{S}_{(\infty)}$-invariant subalgebra $R(\mathbf{x})^{\mathfrak{S}_{(\infty)}}$ of $R(\mathbf{x})$:

$$\Lambda := \{f \in R(\mathbf{x}) : \sigma(f) = f \text{ for all } \sigma \in \mathfrak{S}_{(\infty)}\}$$
$$= \left\{ f = \sum_{\alpha} c_\alpha \mathbf{x}^\alpha \in R(\mathbf{x}) : c_\alpha = c_\beta \text{ if } \alpha, \beta \text{ lie in the same } \mathfrak{S}_{(\infty)}\text{-orbit} \right\}.$$

We refer to the elements of Λ as *symmetric functions* (over $\mathbf{k}$); however, despite this terminology, they are not functions in the usual sense.

Note that Λ is a graded $\mathbf{k}$-algebra, since $\Lambda = \bigoplus_{n\geq 0} \Lambda_n$ where Λ_n are the symmetric functions $f = \sum_\alpha c_\alpha \mathbf{x}^\alpha$ which are *homogeneous of degree* n, meaning $\deg(\mathbf{x}^\alpha) = n$ for all $c_\alpha \neq 0$. A *partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell, 0, 0, \ldots)$ is a weak composition whose entries weakly decrease: $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$. The (uniquely defined) ℓ is said to be the *length* of the partition λ and denoted by $\ell(\lambda)$. Thus, $\ell(\lambda)$ is the number of parts of λ. One sometimes omits trailing zeroes from a partition: e.g., one can write the partition $(3, 1, 0, 0, 0, \ldots)$ as $(3, 1)$. We will often (but not always) write λ_i for the i-th entry of the partition λ (for instance, if $\lambda = (5, 3, 1, 1)$, then $\lambda_2 = 3$ and $\lambda_5 = 0$). If λ_i is nonzero, we will also call it the *i-th part* of λ. The sum $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = \lambda_1 + \lambda_2 + \cdots$ (where $\ell = \ell(\lambda)$) of all entries of λ (or, equivalently, of all parts of λ) is the size $|\lambda|$ of λ. For a given integer n, the partitions of size n are referred to as the *partitions of* n. The empty partition $() = (0, 0, 0, \ldots)$ is denoted by $\varnothing$. Every weak composition α lies in the $\mathfrak{S}_{(\infty)}$-orbit of a unique partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell, 0, 0, \ldots)$ with $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$. For any partition λ, define the *monomial symmetric function*

$$m_\lambda := \sum_{\alpha \in \mathfrak{S}_{(\infty)}\lambda} \mathbf{x}^\alpha. \tag{2.1}$$

Letting λ run through the set *Par* of all partitions, this gives the *monomial* $\mathbf{k}$*-basis* $\{m_\lambda\}$ of Λ. Letting λ run only through the set Par_n of partitions of n gives the monomial $\mathbf{k}$-basis for Λ_n.

It is straightforward to check that $(\Lambda, \underline{m}, \underline{u}, \underline{\Delta}, \underline{\epsilon})$ is a connected graded $\mathbf{k}$-bialgebra of finite type, and hence also a Hopf algebra, where

- The multiplication is the map
$$\Lambda \otimes \Lambda \xrightarrow{\underline{m}} \Lambda,\ m_\mu \otimes m_\nu \mapsto m_\mu m_\nu.$$
- The unit is the inclusion map
$$\mathbf{k} = \Lambda_0 \xrightarrow{\underline{u}} \Lambda.$$
- The comultiplication is the map
$$\Lambda \xrightarrow{\underline{\Delta}} \Lambda \otimes \Lambda,\ m_\lambda \mapsto \sum_{\substack{(\mu,\nu):\\ \mu \sqcup \nu = \lambda}} m_\mu \otimes m_\nu,$$
in which $\mu \sqcup \nu$ is the partition obtained by taking the multiset union of the parts of μ and ν, and then reordering them to make them weakly decreasing.

- The counit is the $\mathbf{k}$-linear map

$$\mathbf{k} = \Lambda_0 \xrightarrow{\epsilon} \Lambda$$

with $\underline{\epsilon}|_{\Lambda_0=\mathbf{k}} = id_{\mathbf{k}}$ and $\underline{\epsilon}|_{I=\bigoplus_{n>0}\Lambda_n} = 0$.

3. Bayer Young Diagram

Definition 3.1. Let λ be a partition.

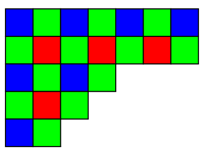
(1) A *colored Young diagram of shape* λ is a Young diagram of shape λ whose cells are colored with green, blue or red.

(2) A Young diagram of shape λ is called a *BGGR-Bayer Young diagram* of shape λ if the corresponding Young diagram of λ has a BGGR pattern. If a tableau does not have enough cells for BGGR pattern (it takes 4 cells to have BGGR), then do it whenever possible. Similarly, *GBRG-Bayer Young diagram*, *GRBG-Bayer Young diagram* and *RGGB-Bayer Young diagram* can be defined.

(3) By *Bayer Young Diagrams*, we will simply mean BGGR-Bayer Young Diagrams (since the other Bayer Young Diagrams can be characterized similarly). Clearly, Bayer Young diagrams are colored Young Diagrams. The converse, however, needs not be true.

(4) Let $\mathcal{YD}$ be the set of all Young diagrams. Let $T : Par \to \mathcal{YD}$ be the bijective map that takes any partition λ to its corresponding Young diagram $T(\lambda)$. Let $\mathcal{BYD}$ be the set of all Bayer Young diagrams. There is a bijective map $\mathcal{B} : Par \to \mathcal{BYD}, \lambda \mapsto \mathcal{B}(\lambda)$.

Example 3.2. Let $\lambda = (7, 7, 4, 3, 2)$. We have



$\mathcal{B}(7, 7, 4, 3, 2)$

Definition 3.3.

(1) Let $\mathcal{B}(\lambda)$ be a Bayer Young diagram of shape λ. Then its corresponding *Bayer Noise Young diagram*, denoted by $\mathcal{C}(\lambda, GBR)$, is the (colored) Young diagram obtained by rearranging the colored cells of $\mathcal{B}(\lambda)$ using the order $G < B < R$ as

follows. First, we rearrange the colored cells of $\mathcal{B}(\lambda)$ to be weakly increasing left-to-right in rows, and then we rearrange the colored cells of the resulting colored Young diagram to be weakly increasing top-to-bottom in columns. One might note that the green part of $\mathcal{C}(\lambda, GBR)$ forms a colored Young subdiagram, denoted by $\mathcal{C}(\lambda, GBR, G)$, of $\mathcal{C}(\lambda, GBR)$ (of shape λ_G) while the region of both the green part and the blue part of $\mathcal{C}(\lambda, GBR)$ forms a colored Young subdiagram, denoted by $\mathcal{C}(\lambda, GBR, GB)$, of $\mathcal{C}(\lambda, GBR)$ (of shape λ_{GB}). Here, λ_G and λ_{GB} are the shapes of the colored Young diagrams $\mathcal{C}(\lambda, GBR, G)$ and $\mathcal{C}(\lambda, GBR, GB)$ respectively. Analogously, one could define $\mathcal{C}(\lambda, GRB)$, $\mathcal{C}(\lambda, RBG)$, $\mathcal{C}(\lambda, RGB)$, $\mathcal{C}(\lambda, BRG)$, $\mathcal{C}(\lambda, BGR)$ and $\mathcal{C}(\lambda, BRG)$. Unless confusion is possible, λ_R always denotes the partition corresponding to the Young subdiagram $\mathcal{C}(\lambda, RGB, R)$ of $\mathcal{C}(\lambda, RGB)$.

(2) Let $\mathfrak{C}$ be the set of all colored Young diagrams and $\mathfrak{A} = \{GBR, GRB, BGR, BRG, RGB, RBG\}$. then $\mathcal{C}$ can be thought of as a map

$$\mathcal{C} : Par \times \mathfrak{A} \to \mathfrak{C},\ (\lambda, E) \mapsto \mathcal{C}(\lambda, E),$$

for any $(\lambda, E) \in Par \times \mathfrak{A}$.

(3) Define a map $\mathfrak{D}_{GBR} : Par \to Par \times Par \times Par,\ \lambda \mapsto (\lambda_{GB}, \lambda_G, \lambda_R)$. If $\lambda, \lambda' \in Par$ with $(\lambda_{GB}, \lambda_G, \lambda_R) = (\lambda'_{GB}, \lambda'_G, \lambda'_R)$, then $\lambda_G = \lambda'_G$, $\lambda_R = \lambda'_R$ and $\lambda_{GB} = \lambda'_{GB}$. This implies that $\mathcal{C}(\lambda, GBR) = \mathcal{C}(\lambda', GBR)$. Since $\lambda_G = \lambda'_G$, $\lambda_R = \lambda'_R$ and $\lambda_{GB} = \lambda'_{GB}$, we have $\lambda_B = \lambda'_B$. Thus, $\mathfrak{D}_{GBR}$ is injective (but not surjective). Composing this map with the projections maps $\pi_{GB} : Par \times Par \times Par \to Par,\ (\lambda_{GB}, \lambda_G, \lambda_R) \mapsto \lambda_{GB}$, $\pi_G : Par \times Par \times Par \to Par,\ (\lambda_{GB}, \lambda_G, \lambda_R) \mapsto \lambda_G$ and $\pi_R : Par \times Par \to Par,\ (\lambda_{GB}, \lambda_G, \lambda_R) \mapsto \lambda_R$, we respectively obtain the maps

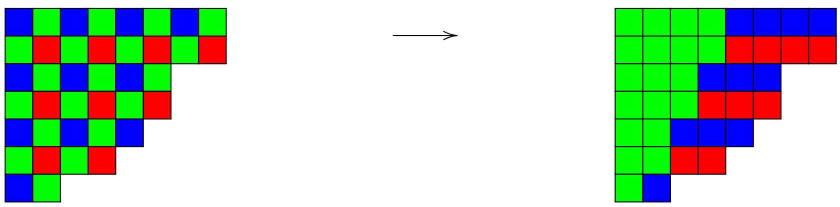
$$\mathfrak{D}_{GB} : Par \to Par,\ \lambda \mapsto \lambda_{GB},$$
$$\mathfrak{D}_G : Par \to Par,\ \lambda \mapsto \lambda_G,$$

and

$$\mathfrak{D}_R : Par \to Par,\ \lambda \mapsto \lambda_R.$$

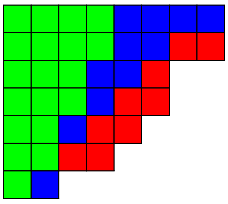
One might note that the maps $\mathfrak{D}_{GB}$ and $\mathfrak{D}_R$ are neither injective nor surjective maps.

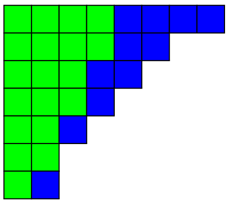
Example 3.4. Consider $\lambda = (8, 8, 6, 6, 5, 4, 2)$. To get $\mathcal{C}_{(\lambda, GBR)}$, we first use the order $G < B < R$ to rearrange $\mathcal{B}(\lambda)$ to be weakly increasing left-to-right in rows. So, we have

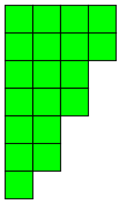


$\mathcal{B}(8, 8, 6, 6, 5, 4, 2)$

Then we rearrange the resulting colored Young diagram to be weakly increasing top-to-bottom in columns to obtain $\mathcal{C}(\lambda, GBR)$. Explicitly, $\mathcal{C}(\lambda, GBR)$ and its corresponding Young subdiagrams $\mathcal{C}(\lambda, GBR, GB)$ and $\mathcal{C}(\lambda, GBR, G)$ are given respectively by the following

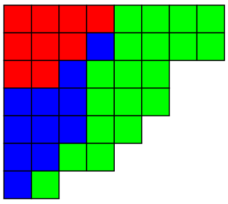


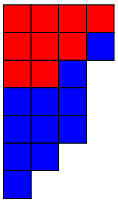


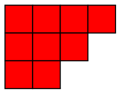


$\mathcal{C}((8,8,6,6,5,4,2), GBR)$ $\mathcal{C}((8,8,6,6,5,4,2), GB)$
$\mathcal{C}((8,8,6,6,5,4,2), G)$

Similarly, one might check that using the order $R < B < G$ gives the following





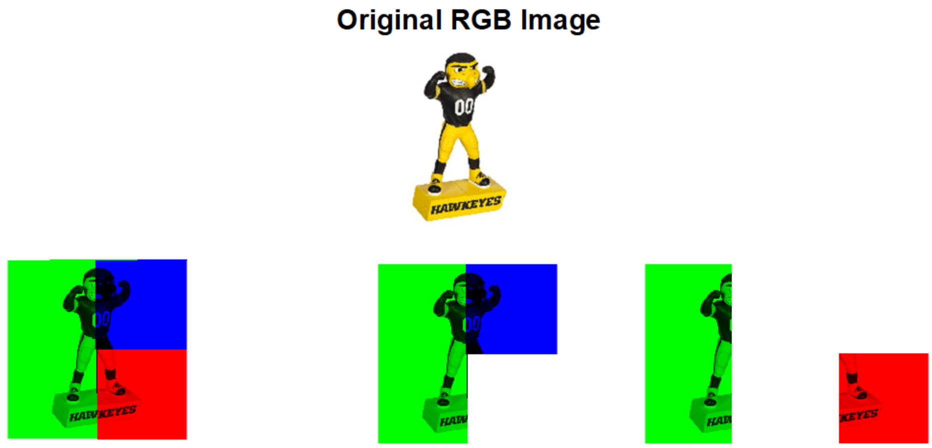


$\mathcal{C}((8,8,6,6,5,4,2), RBG)$ $\mathcal{C}((8,8,6,6,5,4,2), RB)$
$\mathcal{C}((8,8,6,6,5,4,2), R)$

Remark 3.5.

(i) It is well-known that the color channels for a color image are represented by three distinct $2D$ arrays with dimension $m \times n$ for an image with m rows and n columns, with one array for each color, red (color channel 1), green (color channel 2), blue (color channel 3). A pixel color is modeled as 1×3 array [7]. It is also well-known that the spatial domain of each RGB image can be represented as a 3D vector of 2D arrays. The Bayer Noise Young machinery, however, provides us with a new approach by which every Bayer Young diagram can be represented by three special types of colored (noise) diagrams RG, G and R diagrams. This can be depicted in the following example:

3 RGB Channels

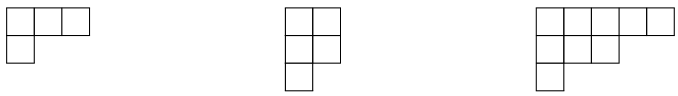


Bayer Noise Young Channels

(ii) Let $\lambda, \lambda' \in Par$ and write $\lambda = (\lambda_1, \cdots, \lambda_m)$ and $\lambda' = (\lambda'_1, \cdots, \lambda'_n)$. Using the convention $\lambda_i = 0$ and $\lambda'_j = 0$ for any $i > m$ and $j > n$, we recall that $\lambda + \lambda'$ is defined as follows:

$$\lambda + \lambda' = (\lambda_1 + \lambda'_1, \cdots, \lambda_k + \lambda'_k),$$

where $k = max\{m, n\}$. For example, if $\lambda = (3, 1)$ and $\lambda' = (2, 2, 1)$, then $\lambda + \lambda' = (5, 3, 1)$. This can be depicted as follows:



$T(\lambda)$ $\qquad$ $T(\lambda')$ $\qquad$ $T(\lambda+\lambda')$

The following proposition is an obvious consequence.

Proposition 3.6.

(1) *The colored Young diagrams* $\mathcal{C}(\lambda, BGR, B)$ *and* $\mathcal{C}(\lambda, BRG, B)$ *have the same shape* λ_B*. Similarly,* $\mathcal{C}(\lambda, GBR, G)$ *and* $\mathcal{C}(\lambda, GRB, G)$ *have the same shape* λ_G *while* $\mathcal{C}(\lambda, RGB, R)$ *and* $\mathcal{C}(\lambda, RBG, R)$ *have the same shape* λ_R*.*

(2) *We have* $\lambda_{GB} = \lambda_{BG}$*,* $\lambda_{GR} = \lambda_{RG}$ *and* $\lambda_{BR} = \lambda_{RB}$*.*

Definition 3.7. Let $\lambda = (\lambda_1, \cdots, \lambda_{\ell(\lambda)}) \in Par$, where $\ell(\lambda)$ is the length of λ.

(1) Define $\lambda^{GB} = (\mu_1, \cdots, \mu_{\ell(\lambda)})$, where

$$\mu_i = \begin{cases} \lambda_i & \text{if } i \text{ is odd} \\ \frac{\lambda_i}{2} & \text{if } i \text{ and } \lambda_i \text{ are both even} \\ \frac{\lambda_i+1}{2} & \text{if } i \text{ is even and } \lambda_i \text{ is odd} \end{cases}$$

(2) We define λ^{G} to be the sequence of nonzero integers $\lambda^{G} = (\mu'_1, \cdots, \mu'_{\ell(\lambda)})$, where

$$\mu'_i = \begin{cases} \frac{\lambda_i}{2} & \text{if } \lambda_i \text{ is even} \\ \frac{\lambda_i-1}{2} & \text{if } i \text{ and } \lambda_i \text{ are both odd} \\ \frac{\lambda_i+1}{2} & \text{if } i \text{ is even and } \lambda_i \text{ is odd} \end{cases}$$

(3) We define λ^{R} to be the sequence of nonzero integers $\lambda^{R} = (\mu''_1, \cdots, \mu''_m)$, where

$$\mu''_i = \begin{cases} \frac{\lambda_{2i}}{2} & \text{if } \lambda_{2i} \text{ is even} \\ \frac{\lambda_{2i}-1}{2} & \text{if } \lambda_{2i} \text{ is odd} \end{cases}$$

and

$$m = \begin{cases} \frac{\ell(\lambda)-1}{2} & \text{if } \ell(\lambda) \text{ is odd} \\ \frac{\ell(\lambda)}{2} & \text{if } \ell(\lambda) \text{ is even} \end{cases}$$

Let $\lambda = (\lambda_1, \cdots, \lambda_{\ell(\lambda)}) \in Par$. Note that λ^{GB}, λ^{G} and λ^{R} need not be in Par. More explicitly, write $\lambda^{G} = (\mu'_1, \cdots, \mu'_{\ell(\lambda)})$ and $\lambda^{R} = (\mu''_1, \cdots, \mu''_m)$. Then if i is odd and $\lambda_i = 1$, then $\mu'_i = 0$. Similarly, if i is even and $\lambda_i = 1$, then $\mu''_i = 0$. The following proposition gives an equivalent setting for Definition (3.3), and the proof is straightforward.

Proposition 3.8. *Let $\lambda = (\lambda_1, \cdots, \lambda_{\ell(\lambda)}) \in Par$, where $\ell(\lambda)$ is the length of λ.*

(1) Let $\tilde{\lambda}^{GB}$ be the partition obtained by reordering the parts of λ^{GB} to make them weakly decreasing. Then we have $\tilde{\lambda}^{GB} = \lambda_{GB}$.

(2) Let $\tilde{\lambda}^{G}$ be the partition obtained by taking the multiset union of the parts of λ^{G}, reordering them to make them weakly decreasing and removing all zero parts of them. Then $\tilde{\lambda}^{G} = \lambda_{G}$.

(3) Let $\tilde{\lambda}^{R}$ be the partition obtained by taking the multiset union of the parts of λ^{R}, reordering them to make them weakly decreasing and removing all zero parts of them. Then $\tilde{\lambda}^{R} = \lambda_{R}$.

Example 3.9. Let $\lambda = (8, 8, 6, 6, 5, 4, 1) \in Par$.

(1) We have $\lambda^{GB} = (8, 4, 6, 3, 5, 2, 1)$ and $\tilde{\lambda}^{GB} = (8, 6, 5, 4, 3, 2, 1) = \lambda_{GB}$.

(2) We have $\lambda^{G} = (4, 4, 3, 3, 2, 2, 0)$ and $\tilde{\lambda}^{G} = (4, 4, 3, 3, 2, 2) = \lambda_{G}$.

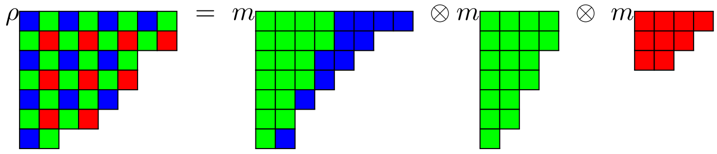
(3) We have $\lambda^{R} = (4, 3, 2) = \tilde{\lambda}^{R} = \lambda_{R}$.

Definition 3.10. For any partition $\lambda \in Par$, the *Bayer Noise monomial* is defined to be the monomial

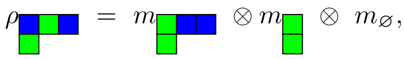
$$\rho_\lambda(x, y, z) = m_{\lambda_{GB}}(x) \otimes m_{\lambda_R}(y) \otimes m_{\lambda_R}(z).$$

We will simply write it as $\rho_\lambda = m_{\lambda_{GB}} \otimes m_{\lambda_G} \otimes m_{\lambda_R}$.

Example 3.11. We have



Remark 3.12. If $\lambda \in Par$, then the partition λ_R could be the empty partition, for example, we have



where $\varnothing$ here is the correspondent empty Young diagram $\mathcal{B}((0))$ of the empty partition (0).

Definition 3.13. Let $\Gamma_n(\mathbf{k})$ be the free $\mathbf{k}$-module with the basis $\{\rho_\lambda\}_{\lambda \in Par_n}$, where Par_n is the set of partitions of n. Note that $dim(\Gamma_n(\mathbf{k})) = |Par_n|$, where $|Par_n|$ is the number of elements of Par_n. Let $\Gamma(\mathbf{k}) = \bigoplus_{n \geq 0} \Gamma_n(\mathbf{k})$. Then the set $\{\rho_\lambda\}_{\lambda \in Par}$ forms a basis for $\Gamma(\mathbf{k})$ over $\mathbf{k}$, and $\Gamma(\mathbf{k})$ is called the *Bayer Noise module*. Obviously, as modules, $\Gamma_n(\mathbf{k}) \cong \Lambda_n(\mathbf{k})$ for every $n \in \mathbb{N}$ and hence $\Gamma(\mathbf{k}) \cong \Lambda(\mathbf{k})$.

Remark 3.14.

(1) When no confusion is possible, we will simply write Γ_n and Γ instead of $\Gamma_n(\mathbf{k})$ and $\Gamma(\mathbf{k})$ respectively.

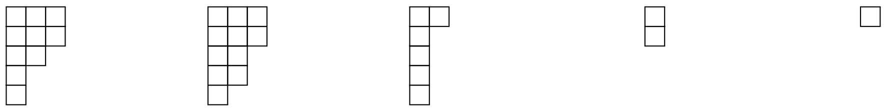
(2) Let $\mu, \nu \in Par$. Then, in general, $(\mu_{GB} + \nu_{GB}, \mu_G + \nu_G, \mu_R + \nu_R)$ need not be in $\mathfrak{D}_{GBR}(Par)$, and hence $\rho_\mu \, \rho_\nu$ need not be in Γ, where $\rho_\mu \, \rho_\nu$ is the regular multiplication of the monomials ρ_μ and ρ_ν. For example, if $\mu = (1,1) = \nu$, then $\mu_{GB} = \nu_{GB} = (1,1)$, $\mu_G = \nu_G = (1)$ and $\mu_R = \nu_R = (0)$ (the empty partition). However, $(\mu_{GB} + \nu_{GB}, \mu_G + \nu_G, \mu_R + \nu_R) = ((2,2),(2),(0))$ which is clearly not in $\mathfrak{D}_{GBR}(Par)$. It turns out that the operation $(\rho_\mu, \rho_\nu) \mapsto \rho_\mu \, \rho_\nu$ does not define an algebra structure on Γ.

(3) One might notice that in general if $(\mu, \nu) \in Par \times Par$, then $((\mu \sqcup \nu)_{GB}, (\mu \sqcup \nu)_G, (\mu \sqcup \nu)_R) \neq (\mu_{GB} \sqcup \nu_{GB}, \mu_R \sqcup \nu_R)$ and $(\mu_{GB} \sqcup \nu_{GB}, \mu_G \sqcup \nu_G, \mu_R \sqcup \nu_R)$ need not be in $\mathfrak{D}_{GBR}(Par)$. For example, if $\mu = (3,3,2)$, $\nu = (3,1)$, then we have

$\mathcal{B}(\mu)$ $\qquad$ $\mathcal{C}(\mu, GB)$ $\qquad$ $\mathcal{C}(\mu, G)$ $\qquad$ $\mathcal{C}(\mu, R)$

$\mathcal{B}(\nu)$ $\qquad$ $\mathcal{C}(\nu, GB)$ $\qquad$ $\mathcal{C}(\nu, G)$ $\qquad$ $\mathcal{C}(\nu, R)$

$\mathcal{B}((\mu \sqcup \nu))$ $\qquad$ $\mathcal{C}((\mu \sqcup \nu), GB)$ $\qquad$ $\mathcal{C}((\mu \sqcup \nu), G)$ $\qquad$ $\mathcal{C}((\mu \sqcup \nu), R)$



$T((\mu \sqcup \nu)_{GB})$ $\quad T(\mu_{GB} \sqcup \nu_{GB})$ $\quad T(\mu_G \sqcup \nu_G)$ $\quad T((\mu \sqcup \nu)_R)$ $\quad T(\mu_R \sqcup \nu_R)$

Thus, $((\mu \sqcup \nu)_{GB}, (\mu \sqcup \nu)_G), (\mu \sqcup \nu)_R) \neq (\mu_{GB} \sqcup \nu_{GB}, \mu_G \sqcup \nu_G, \mu_R \sqcup \nu_R)$ and $(\mu_{GB} \sqcup \nu_{GB}, \mu_G \sqcup \nu_G, \mu_R \sqcup \nu_R) \notin \mathfrak{D}_{GBR}(Par)$.

4. Algebraic Structures

Recall that for any $n, m_1, \cdots, m_t \in \mathbb{N}$ with $\sum_{i=1}^{t} m_i = n$ and $t \geq 2$, the multinomial coefficient, denoted by $\binom{n}{m_1, \cdots, m_t}$, is defined by

$$\binom{n}{m_1, \cdots, m_t} = \frac{(\sum_{i=1}^{t} m_i)!}{(m_1)! \cdots (m_t)!}$$

Let η be the map

$$\eta : \Gamma \otimes \Gamma \to \Gamma, \ \ \rho_\lambda \otimes \rho_{\lambda'} \ \mapsto \binom{|\lambda| + |\lambda'|}{|\lambda|, |\lambda'|} \rho_{\lambda \sqcup \lambda'},$$

and let

$$\mathbf{k} = \Gamma_0 \xrightarrow{u} \Gamma$$

be the inclusion map. We have the following proposition.

Proposition 4.1. *The triple* (Γ, η, u) *is a* $\mathbf{k}$*-algebra.*

Proof. Consider the following diagrams:

$$\begin{tikzcd} \Gamma \otimes \Gamma \otimes \Gamma \arrow[r, "id \otimes \eta"] \arrow[d, "\eta \otimes id"'] & \Gamma \otimes \Gamma \arrow[d, "\eta"] \\ \Gamma \otimes \Gamma \arrow[r, "\eta"'] & \Gamma \end{tikzcd} \qquad \begin{tikzcd} \Gamma \otimes \mathbf{k} \arrow[d, "id \otimes u"'] & \Gamma \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "id"'] & \mathbf{k} \otimes \Gamma \arrow[d, "u \otimes id"] \\ \Gamma \otimes \Gamma \arrow[r, "\eta"] & \Gamma & \Gamma \otimes \Gamma \arrow[l, "\eta"'] \end{tikzcd} \tag{4.1}$$

We have to show that they are commutative. Write $\rho_\lambda \odot \rho_{\lambda'} \;=\; \eta(\rho_\lambda \otimes \rho_{\lambda'})$.

$$\begin{aligned}
(\rho_\lambda \odot \rho_{\lambda'}) \odot \rho_{\lambda''} &= (\binom{|\lambda|+|\lambda'|}{|\lambda|,|\lambda'|}\ \rho_{\lambda\sqcup\lambda'}) \odot \rho_{\lambda''}\\
&= \binom{|\lambda|+|\lambda'|}{|\lambda|,|\lambda'|}\ (\rho_{\lambda\sqcup\lambda'} \odot \rho_{\lambda''})\\
&= \binom{|\lambda|+|\lambda'|}{|\lambda|,|\lambda'|}\binom{|\lambda\sqcup\lambda'|+|\lambda''|}{|\lambda\sqcup\lambda'|,|\lambda''|}\ \rho_{\lambda\sqcup\lambda'\sqcup\lambda''}\\
&= \binom{|\lambda|+|\lambda'|}{|\lambda|,|\lambda'|}\binom{|\lambda|+|\lambda'|+|\lambda''|}{|\lambda|+|\lambda'|,|\lambda''|}\ \rho_{\lambda\sqcup\lambda'\sqcup\lambda''}\\
&= \frac{(|\lambda|+|\lambda'|+|\lambda''|)!}{(|\lambda|)!\ (|\lambda'|)!\ (|\lambda''|)!}\ \rho_{\lambda\sqcup\lambda'\sqcup\lambda''}\\
&= \binom{|\lambda|+|\lambda'|+|\lambda''|}{|\lambda|,|\lambda'|,|\lambda''|}\ \rho_{\lambda\sqcup\lambda'\sqcup\lambda''}
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\rho_\lambda \odot (\rho_{\lambda'} \odot \rho_{\lambda''}) &= \rho_\lambda\ \odot\ (\binom{|\lambda'|+|\lambda''|}{|\lambda'|,|\lambda''|}\ \rho_{\lambda'\sqcup\lambda''})\\
&= \binom{|\lambda'|+|\lambda''|}{|\lambda'|,|\lambda''|}\ (\rho_\lambda\ \odot\ \rho_{\lambda'\sqcup\lambda''})\\
&= \binom{|\lambda'|+|\lambda''|}{|\lambda'|,|\lambda''|}\ (\binom{|\lambda|+|\lambda'\sqcup\lambda''|}{|\lambda|,|\lambda'\sqcup\lambda''|}\ \rho_{\lambda\sqcup(\lambda'\sqcup\lambda'')}\\
&= \binom{|\lambda'|+|\lambda''|}{|\lambda'|,|\lambda''|}\ (\binom{|\lambda|+|\lambda'|+|\lambda''|}{|\lambda|,|\lambda'|+|\lambda''|}\ \rho_{\lambda\sqcup\lambda'\sqcup\lambda''}\\
&= \binom{|\lambda|+|\lambda'|+|\lambda''|}{|\lambda|,|\lambda'|,|\lambda''|}\ \rho_{\lambda\sqcup\lambda'\sqcup\lambda''}.
\end{aligned}$$

Accordingly, we have $(\rho_\lambda \odot \rho_{\lambda'}) \odot \rho_{\lambda''} \;=\; \rho_\lambda \odot (\rho_{\lambda'} \odot \rho_{\lambda''})$ for any $\lambda, \lambda', \lambda'' \in Par$, and hence the commutativity of the first diagram (the associativity diagram) of (4.1) follows. For the other diagram, we note that

$$\begin{aligned}
\eta(u \otimes id)(\rho_\lambda \otimes 1) &= \eta(\rho_\lambda \otimes 1)\\
&= \binom{|\lambda|+|\emptyset|}{|\lambda|,|\emptyset|}\ \rho_{\lambda\sqcup\emptyset}\\
&= \binom{|\lambda|+0}{|\lambda|,0}\ \rho_\lambda\\
&= \binom{|\lambda|}{|\lambda|,0}\ \rho_\lambda
\end{aligned}$$

$$
\begin{aligned}
&= \rho_\lambda \\
&= id(\rho_\lambda) \\
&= \binom{0+|\lambda|}{0,|\lambda|}\rho_\lambda \\
&= \binom{|\emptyset|+|\lambda|}{|\emptyset|,|\lambda|}\rho_{\emptyset\sqcup\lambda} \\
&= \eta(1\otimes\rho_\lambda) \\
&= \eta(id\otimes u)(1\otimes\rho_\lambda)
\end{aligned}
$$

As a consequence, (Γ,η,ϵ) is a **k**-algebra. □

Definition 4.2. Let $Par^e = \{\lambda \in Par : \text{ all } \lambda\text{- parts are even}\}$, and let $\Gamma^{(e,n)}(\mathbf{k})$ be the free **k**-module with the basis $\{\rho_\lambda\}_{\lambda\in Par^{(e,n)}}$, where $Par^{(e,n)} = Par_n \bigcap Par^e$. Let $\Gamma^e(\mathbf{k}) = \bigoplus_{n\geq 0}\Gamma^{(e,n)}(\mathbf{k})$. Then the set $\{\rho_\lambda\}_{\lambda\in Par^e}$ forms a basis for $\Gamma^e(\mathbf{k})$ over **k**.

The proof of the following lemma is straightforward and left to the reader.

Lemma 4.3. *We have $(\lambda+\lambda')_{GB} = \lambda_{GB}+\lambda'_{GB}$ and $(\lambda+\lambda')_R = \lambda_R+\lambda'_R$ for every $\lambda,\lambda' \in Par^e$.*

The following theorem emphasizes the importance of Definition (4.2)

Theorem 4.4. *We have the following:*

(1) $\binom{|(\lambda+\lambda')_{GB}|}{|\lambda_{GB}|} = \binom{|\lambda_{GB}|+|\lambda'_{GB}|}{|\lambda_{GB}|} = \binom{|\lambda_{GB}|+|\lambda'_{GB}|}{|\lambda_{GB}|,|\lambda'_{GB}|}$ *for any $\lambda,\lambda' \in Par^e$.*

(2) $\binom{|(2(\lambda+\lambda'))_{GB}|}{|2\lambda_{GB}|} = \binom{2(|\lambda_{GB}|+|\lambda'_{GB}|)}{2|\lambda_{GB}|} = \binom{2(|\lambda_{GB}|+|\lambda'_{GB}|)}{2|\lambda'_{GB}|} = \binom{2(|\lambda_{GB}|+|\lambda'_{GB}|)}{2|\lambda_{GB}|,2|\lambda'_{GB}|}$ *for any $\lambda,\lambda' \in Par$.*

(3) $\binom{|(\lambda+\lambda')_{GB}|}{|\lambda_{GB}|}\binom{|(\lambda+\lambda'+\lambda'')_{GB}|}{|(\lambda+\lambda')_{GB}|} = \binom{|\lambda_{GB}|+|\lambda'_{GB}|+|\lambda''_{GB}|}{|\lambda_{GB}|,|\lambda'_{GB}|,|\lambda''_{GB}|}$ *for any $\lambda,\lambda',\lambda'' \in Par^e$.*

(4) *In general, we have*

$$\binom{|(\lambda^{(1)}+\lambda^{(2)}+\ldots+\lambda^{(t)})_{GB}|}{|\lambda^{(1)}_{GB}|,|\lambda^{(2)}_{GB}|,\ldots,|\lambda^{(t)}_{GB}|} = \binom{|\lambda^{(1)}_{GB}|+|\lambda^{(2)}_{GB}|+\ldots+|\lambda^{(t)}_{GB}|}{|\lambda^{(1)}_{GB}|,|\lambda^{(2)}_{GB}|,\ldots,|\lambda^{(t)}_{GB}|}$$

for every $(\lambda^{(1)},\lambda^{(2)},...,\lambda^{(t)}) \in Par^e$, and

$$\binom{|(2(\lambda^{(1)}+\lambda^{(2)}+\ldots+\lambda^{(t)}))_{GB}|}{2|\lambda^{(1)}_{GB}|,2|\lambda^{(2)}_{GB}|,\ldots,2|\lambda^{(t)}_{GB}|} = \binom{2|\lambda^{(1)}_{GB}|+2|\lambda^{(2)}_{GB}|+\ldots+2|\lambda^{(t)}_{GB}|}{2|\lambda^{(1)}_{GB}|,2|\lambda^{(2)}_{GB}|,\ldots,2|\lambda^{(t)}_{GB}|}$$

for every $(\lambda^{(1)},\lambda^{(2)},...,\lambda^{(t)}) \in Par$.

(5) The triple $(\Gamma^e(\mathbf{k}), \eta_e, u_e)$ *is a* $\mathbf{k}$*-algebra, where* η_e *is the map*

$$\eta_e \,:\, \Gamma^e(\mathbf{k}) \otimes \Gamma^e(\mathbf{k}) \to \Gamma^e(\mathbf{k}),\;\; \rho_\lambda \otimes \rho_{\lambda'} \;\mapsto\; \binom{|(\lambda+\lambda')_{GB}|}{|\lambda_{GB}|} \rho_{\lambda \sqcup \lambda'}$$

and

$$\mathbf{k} = \Gamma^{(e,0)}(\mathbf{k}) \xrightarrow{u_e} \Gamma^e(\mathbf{k})$$

is the inclusion map.

Proof. (1) We have

$$\begin{aligned}
\binom{|(\lambda+\lambda')_{GB}|}{|\lambda_{GB}|} &= \binom{|\lambda_{GB}+\lambda'_{GB}|}{|\lambda_{GB}|} \text{ (by using Lemma (4.3))}\\
&= \binom{|\lambda_{GB}|+|\lambda'_{GB}|}{|\lambda_{GB}|}\\
&= \binom{|\lambda_{GB}|+|\lambda'_{GB}|}{|\lambda_{GB}|,\, |\lambda'_{GB}|}.
\end{aligned}$$

(2) An easy calculation gives the following:

$$\begin{aligned}
\binom{|(2(\lambda+\lambda'))_{GB}|}{|2\lambda_{GB}|} &= \binom{2|(\lambda+\lambda')_{GB}|}{2|\lambda_{GB}|} \text{ (since } |2\lambda| = 2|\lambda|,\ \forall \lambda \in Par)\\
&= \binom{2(|\lambda_{GB}|+|\lambda'_{GB}|)}{2|\lambda_{GB}|} \text{ (by using Lemma (4.3))}\\
&= \binom{2(|\lambda_{GB}|+|\lambda'_{GB}|)}{2|\lambda'_{GB}|}\\
&= \binom{2(|\lambda_{GB}|+|\lambda'_{GB}|)}{2|\lambda_{GB}|,\, 2|\lambda'_{GB}|}.
\end{aligned}$$

(3) We calculate

$$\begin{aligned}
\binom{|(\lambda+\lambda')_{GB}|}{|\lambda_{GB}|}\binom{|(\lambda+\lambda'+\lambda'')_{GB}|}{|(\lambda+\lambda')_{GB}|} &= \binom{|\lambda_{GB}|+|\lambda'_{GB}|}{|\lambda_{GB}|}\binom{|\lambda_{GB}|+|\lambda'_{GB}|+|\lambda''_{GB}|}{|\lambda_{GB}|+|\lambda'_{GB}|}\\
&= \frac{(|\lambda_{GB}|+|\lambda'_{GB}|)!}{(|\lambda_{GB}|)!(|\lambda'_{GB}|)!}\,\frac{(|\lambda_{GB}|+|\lambda'_{GB}|+|\lambda''_{GB}|)!}{(|\lambda_{GB}|+|\lambda'_{GB}|)!(|\lambda''_{GB}|)!}\\
&= \frac{(|\lambda_{GB}|+|\lambda'_{GB}|+|\lambda''_{GB}|)!}{(|\lambda_{GB}|)!(|\lambda'_{GB}|)!(|\lambda''_{GB}|)!}\\
&= \binom{|\lambda_{GB}|+|\lambda'_{GB}|+|\lambda''_{GB}|}{|\lambda_{GB}|, |\lambda'_{GB}|, |\lambda''_{GB}|}.
\end{aligned}$$

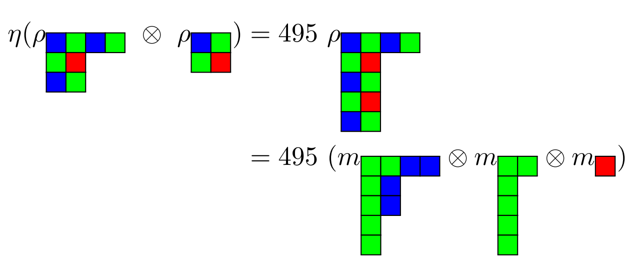
(4) This follows immediately from the proof of the previous part.

(5) This can be easily proved using parts (ii) and (iii) of the proposition.

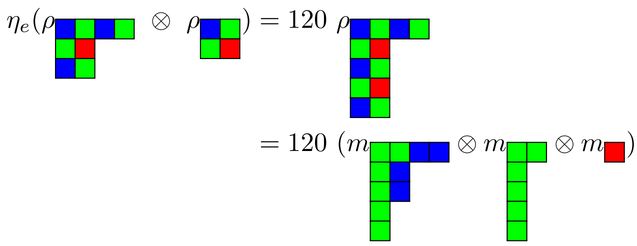
□

Example 4.5.

(1) Let $\lambda = (4, 2, 2)$ and $\lambda' = (2, 2)$. Then we have



(2) A direct calculation gives the following:



5. Coalgebraic Structures

Consider the map

$$\Delta \rho_\lambda = \sum_{\substack{(\mu,\nu)\in Par\times Par: \\ \mu\sqcup\nu=\lambda}} \rho_\mu \otimes \rho_\nu, \tag{5.1}$$

in which $\mu \sqcup \nu$ is the partition obtained by taking the multiset union of the parts of μ and ν, and then reordering them to make them weakly decreasing. Interestingly, one might define the map $\tilde{\Delta} : \Gamma \to \Gamma \otimes \Gamma$ defined **k**-linearly by

$$\tilde{\Delta} \rho_\lambda = \sum_{\substack{(\mu,\nu)\in Par\times Par: \\ \mu_U\sqcup\nu_U=\lambda_U, \forall U\in\{GB,R\}}} \binom{|\mu| + |\nu|}{|\mu|, |\nu|} \rho_\mu \otimes \rho_\nu, \tag{5.2}$$

in which $\mu_U \sqcup \nu_U$ is the partition obtained by taking the multiset union of the parts of μ_U and ν_U, and then reordering them to make them weakly decreasing. From image processing point of view, we find the image noise corresponding to the Bayer Noise Young diagram of λ, and then we split the resulting one into pieces: one with less noise having only two color sensors (G and B), and one with more noise having only one color sensor R. We have the following theorem.

Theorem 5.1. *Let* $\Gamma \xrightarrow{\epsilon} \mathbf{k}$ *be the map defined* $\mathbf{k}$*-linearly by*

$$\epsilon|_{\Gamma_0=\mathbf{k}} = id_{\mathbf{k}} \text{ and } \epsilon|_{I=\bigoplus_{n>0}\Gamma_n} = 0.$$

Then

(i) The triple $(\Gamma, \Delta, \epsilon)$ *is a* $\mathbf{k}$*-coalgebra.*

(ii) The triple $(\Gamma, \tilde{\Delta}, \epsilon)$ *is a* $\mathbf{k}$*-coalgebra.*

Proof. The proof of (i) is obvious. To prove part (ii), we have to show the following diagrams are commutative.

$$\begin{tikzcd} & \Gamma\otimes\Gamma\otimes\Gamma & \\ \Gamma\otimes\Gamma \arrow[ur, "\tilde{\Delta}\otimes id"] & & \Gamma\otimes\Gamma \arrow[ul, "id\otimes\tilde{\Delta}"'] \\ & \Gamma \arrow[ul, "\tilde{\Delta}"] \arrow[ur, "\tilde{\Delta}"'] & \end{tikzcd} \qquad \begin{tikzcd} \Gamma\otimes\mathbf{k} \arrow[r, "\Phi"] & \Gamma & \mathbf{k}\otimes\Gamma \arrow[l, "\Psi"'] \\ \Gamma\otimes\Gamma \arrow[u, "id\otimes\epsilon"] & \Gamma \arrow[l, "\tilde{\Delta}"] \arrow[r, "\tilde{\Delta}"'] \arrow[u, "id"] & \Gamma\otimes\Gamma \arrow[u, "\epsilon\otimes id"'] \end{tikzcd} \tag{5.3}$$

Here Φ and Ψ are the isomorphisms $\Phi : \Gamma\otimes\mathbf{k} \to \Gamma,\ \rho_\lambda\otimes 1 \mapsto \rho_\lambda$ and $\Psi : \mathbf{k}\otimes\Gamma \to \Gamma,\ 1\otimes\rho_\lambda \mapsto \rho_\lambda$. For any $\lambda \in Par$, we have

$$\begin{aligned}
(\tilde{\Delta}\otimes id)\tilde{\Delta}\rho_\lambda &= (\tilde{\Delta}\otimes id)\Big(\sum_{\substack{(\mu,\mu')\in Par\times Par:\\ \mu_U\sqcup\mu'_U=\lambda_U,\forall U\in\{GB,R\}}} \binom{|\mu|+|\mu'|}{|\mu|,|\mu'|}\rho_\mu\otimes\rho_{\mu'}\Big) \\
&= \sum_{\substack{(\mu,\mu')\in Par\times Par:\\ \mu_U\sqcup\mu'_U=\lambda_U,\forall U\in\{GB,R\}}} \binom{|\mu|+|\mu'|}{|\mu|,|\mu'|}\tilde{\Delta}\rho_\mu\otimes\rho_{\mu'} \\
&= \sum_{\substack{(\mu,\mu')\in Par\times Par:\\ \mu_U\sqcup\mu'_U=\lambda_U,\forall U\in\{GB,R\}}} \binom{|\mu|+|\mu'|}{|\mu|,|\mu'|} \sum_{\substack{(\nu,\nu')\in Par\times Par:\\ \nu_W\sqcup\nu'_W=\mu_W,\forall W\in\{GB,R\}}} \binom{|\nu|+|\nu'|}{|\nu|,|\nu'|}(\rho_\nu\otimes\rho_{\nu'})\otimes\rho_{\mu'} \\
&= \sum_{\substack{(\nu,\nu',\mu')\in Par\times Par\times Par:\\ \nu_U\sqcup\nu'_U\sqcup\mu'_U=\lambda_U,\forall U\in\{GB,R\}}} \binom{|\mu|+|\mu'|}{|\mu|,|\mu'|}\binom{|\nu|+|\nu'|}{|\nu|,|\nu'|}\rho_\nu\otimes\rho_{\nu'}\otimes\rho_{\mu'}
\end{aligned}$$

One can easily check the following:

$$|\lambda| = |\lambda_{GB}| + |\lambda_R| = |\mu_{GB}| + |\mu'_{GB}| + |\mu_R| + |\mu'_R| = |\mu| + |\mu'|$$

and

$$|\mu| = |\mu_{GB}| + |\mu_R| = |\nu_{GB}| + |\nu'_{GB}| + |\nu_R| + |\nu'_R| = |\nu| + |\nu'|.$$

As a result, we have

$$\begin{aligned}\binom{|\mu| + |\mu'|}{|\mu|, |\mu'|} \binom{|\nu| + |\nu'|}{|\nu|, |\nu'|} &= \binom{|\nu| + |\nu'| + |\mu'|}{|\mu|, |\mu'|} \binom{|\mu|}{|\nu|, |\nu'|} \\ &= \binom{|\nu| + |\nu'| + |\mu'|}{|\nu|, |\nu'|, |\mu'|} \text{ (by part (2) of Proposition (4.1)).}\end{aligned}$$

Thus, we have

$$\begin{aligned}(\tilde{\Delta} \otimes id)\tilde{\Delta}\rho_\lambda &= \sum_{\substack{(\nu,\nu',\mu') \in Par \times Par \times Par: \\ \nu_U \sqcup \nu'_U \sqcup \mu'_U = \lambda_U, \forall U \in \{GB,R\}}} \binom{|\nu| + |\nu'| + |\mu'|}{|\nu|, |\nu'|, |\mu'|} \rho_\nu \otimes \rho_{\nu'} \otimes \rho_{\mu'} \\ &= \sum_{\substack{(\mu,\mu') \in Par \times Par: \\ \mu_U \sqcup \mu'_U = \lambda_U, \forall U \in \{GB,R\}}} \binom{|\mu| + |\mu'|}{|\mu|, |\mu'|} \sum_{\substack{(\nu,\nu') \in Par \times Par: \\ \nu_W \sqcup \nu'_W = \mu'_W, \forall W \in \{GB,R\}}} \binom{|\nu| + |\nu'|}{|\nu|, |\nu'|} \rho_\mu \otimes (\rho_\nu \otimes \rho_{\nu'}) \\ &= \sum_{\substack{(\mu,\mu') \in Par \times Par: \\ \mu_U \sqcup \mu'_U = \lambda_U, \forall U \in \{GB,R\}}} \binom{|\mu| + |\mu'|}{|\mu|, |\mu'|} \rho_\mu \otimes \sum_{\substack{(\nu,\nu') \in Par \times Par: \\ \nu_W \sqcup \nu'_W = \mu'_W, \forall W \in \{GB,R\}}} \binom{|\nu| + |\nu'|}{|\nu|, |\nu'|} (\rho_\nu \otimes \rho_{\nu'}) \\ &= \sum_{\substack{(\mu,\mu') \in Par \times Par: \\ \mu_U \sqcup \mu'_U = \lambda_U, \forall U \in \{GB,R\}}} \binom{|\mu| + |\mu'|}{|\mu|, |\mu'|} \rho_\mu \otimes \tilde{\Delta}\rho_{\mu'} \\ &= (id \otimes \tilde{\Delta})(\sum_{\substack{(\mu,\mu') \in Par \times Par: \\ \mu_U \sqcup \mu'_U = \lambda_U, \forall U \in \{GB,R\}}} \binom{|\mu| + |\mu'|}{|\mu|, |\mu'|} \rho_\mu \otimes \rho_{\mu'}) \\ &= (id \otimes \tilde{\Delta})\tilde{\Delta}\rho_\lambda.\end{aligned}$$

Therefore, the commutativity of the associativity diagram follows. Checking the commutativity of the unity diagram can be done as follows:

$$\begin{aligned}
\Psi(\epsilon\otimes id)\tilde{\Delta}\rho_\lambda &= \Psi(\epsilon\otimes id)\Big(\sum_{\substack{(\mu,\mu')\in Par\times Par:\\ \mu_U\sqcup\mu'_U=\lambda_U,\forall U\in\{GB,R\}}}\binom{|\mu|+|\mu'|}{|\mu|,|\mu'|}\rho_\mu\otimes\rho_{\mu'}\Big)\\
&= \Psi\Big(\sum_{\substack{(\mu,\mu')\in Par\times Par:\\ \mu_U\sqcup\mu'_U=\lambda_U,\forall U\in\{GB,R\}}}\binom{|\mu|+|\mu'|}{|\mu|,|\mu'|}\epsilon(\rho_\mu)\otimes\rho_{\mu'}\Big)\\
&= \sum_{\substack{(\mu,\mu')\in Par\times Par:\\ \mu_U\sqcup\mu'_U=\lambda_U,\forall U\in\{GB,R\}}}\binom{|\mu|+|\mu'|}{|\mu|,|\mu'|}\epsilon(\rho_\mu)\,\rho_{\mu'}\\
&= \rho_\lambda\ (\text{ since } \epsilon|_{\mathbf{k}} = id_{\mathbf{k}} \text{ and } \epsilon|_{I=\bigoplus_{n>0}\Gamma_n}=0).\\
&= id(\rho_\lambda)\\
&= \sum_{\substack{(\mu,\mu')\in Par\times Par:\\ \mu_U\sqcup\mu'_U=\lambda_U,\forall U\in\{GB,R\}}}\binom{|\mu|+|\mu'|}{|\mu|,|\mu'|}\rho_\mu\,\epsilon(\rho_{\mu'})\\
&= \Phi\Big(\sum_{\substack{(\mu,\mu')\in Par\times Par:\\ \mu_U\sqcup\mu'_U=\lambda_U,\forall U\in\{GB,R\}}}\binom{|\mu|+|\mu'|}{|\mu|,|\mu'|}\rho_\mu\otimes\epsilon(\rho_{\mu'})\Big)\\
&= \Phi(id\otimes\epsilon)\Big(\sum_{\substack{(\mu,\mu')\in Par\times Par:\\ \mu_U\sqcup\mu'_U=\lambda_U,\forall U\in\{GB,R\}}}\binom{|\mu|+|\mu'|}{|\mu|,|\mu'|}\rho_\mu\otimes\rho_{\mu'}\Big)\\
&= \Phi(id\otimes\epsilon)\tilde{\Delta}\rho_\lambda.
\end{aligned}$$

It follows that $(\Gamma,\tilde{\Delta},\epsilon)$ is a $\mathbf{k}$-coalgebra. □

We call the $\mathbf{k}$-coalgebra $(\Gamma,\tilde{\Delta},\epsilon)$ as the *Bayer Noise coalgebra* over $\mathbf{k}$. The following proposition gives an explicit description for primitive with respect to the comultiplication $\tilde{\Delta}$.

Proposition 5.2. *Let $\lambda\in Par$. The element ρ_λ is primitive (with respect to $\tilde{\Delta}$) if and only if $\lambda=(m)$ for some non-negative integer m.*

Proof. It is straightforward to prove that $\tilde{\Delta}\rho_\lambda=\rho_\lambda\otimes 1+1\otimes\rho_\lambda$ if and only if $\lambda=(m)$ for some non-negative integer m. This completes the proof. □

Let $\widehat{\Delta}:\Gamma^e(\mathbf{k})\to\Gamma^e(\mathbf{k})\otimes\Gamma^e(\mathbf{k})$ be the map defined $\mathbf{k}$-linearly by

$$\widehat{\Delta}\rho_\lambda = \sum_{\substack{(\mu,\nu)\in Par^e\times Par^e:\\ \mu_U\sqcup\nu_U=\lambda_U,\forall U\in\{GB,R\}}}\binom{|\mu|+|\nu|}{|\mu|,|\nu|}\rho_\mu\otimes\rho_\nu, \tag{5.4}$$

in which $\mu_U \sqcup \nu_U$ is the partition obtained by taking the multiset union of the parts of μ_U and ν_U, and then reordering them to make them weakly decreasing.

Using part (3) of Theorem (4.4), the following theorem can be proved similarly to the proof of Theorem (5.1).

Theorem 5.3. *The triple* $(\Gamma^e(\mathbf{k}), \widehat{\Delta}, \widehat{\epsilon})$ *is a* $\mathbf{k}$*-coalgebra, where* $\Gamma \stackrel{\widehat{\epsilon}}{\to} \mathbf{k}$ *is the map defined* $\mathbf{k}$*-linearly by*

$$\widehat{\epsilon}|_{\Gamma^{(e,0)}=\mathbf{k}} = id_{\mathbf{k}} \text{ and } \widehat{\epsilon}|_{I=\bigoplus_{n>0}\Gamma^{(e,n)}} = 0.$$

The primitive elements in $\Gamma^e(\mathbf{k})$ (with respect to the comultiplication $\widehat{\Delta}$) can be explicitly described as follows:

Proposition 5.4. *The primitive basis elements for* $\Gamma^e(\mathbf{k})$ *(with respect to the comultiplication* $\widehat{\Delta}$*) are precisely of the form* ρ_λ*, where* $\lambda = (m)$ *for some* $m \in 2\mathbb{N} = \{0, 2, 4, \ldots\}$.

Proof. The proof is very similar to the proof of Proposition (5.2). □

Similarly, we define the map $\Delta^{(e)} : \Gamma \to \Gamma \otimes \Gamma$ defined $\mathbf{k}$-linearly by

$$\Delta^{(e)}\rho_\lambda = \sum_{\substack{(\mu,\nu)\in Par\times Par: \\ \mu_U \sqcup \nu_U = \lambda_U, \forall U \in \{GB,R\}}} \binom{2|\mu| + 2|\nu|}{2|\mu|, 2|\nu|} \rho_\mu \otimes \rho_\nu, \tag{5.5}$$

in which $\mu_U \sqcup \nu_U$ is the partition obtained by taking the multiset union of the parts of μ_U and ν_U, and then reordering them to make them weakly decreasing.
The following are analogous consequences to those of Theorem (5.3) and Proposition (5.4) respectively.

Theorem 5.5. *The triple* $(\Gamma, \Delta^{(e)}, \epsilon^{(e)})$ *is a* $\mathbf{k}$*-coalgebra, where* $\Gamma \stackrel{\epsilon^{(e)}}{\to} \mathbf{k}$ *is the map defined* $\mathbf{k}$*-linearly by*

$$\epsilon^{(e)}|_{\Gamma_0=\mathbf{k}} = id_{\mathbf{k}} \text{ and } \epsilon^{(e)}|_{I=\bigoplus_{n>0}\Gamma_n} = 0.$$

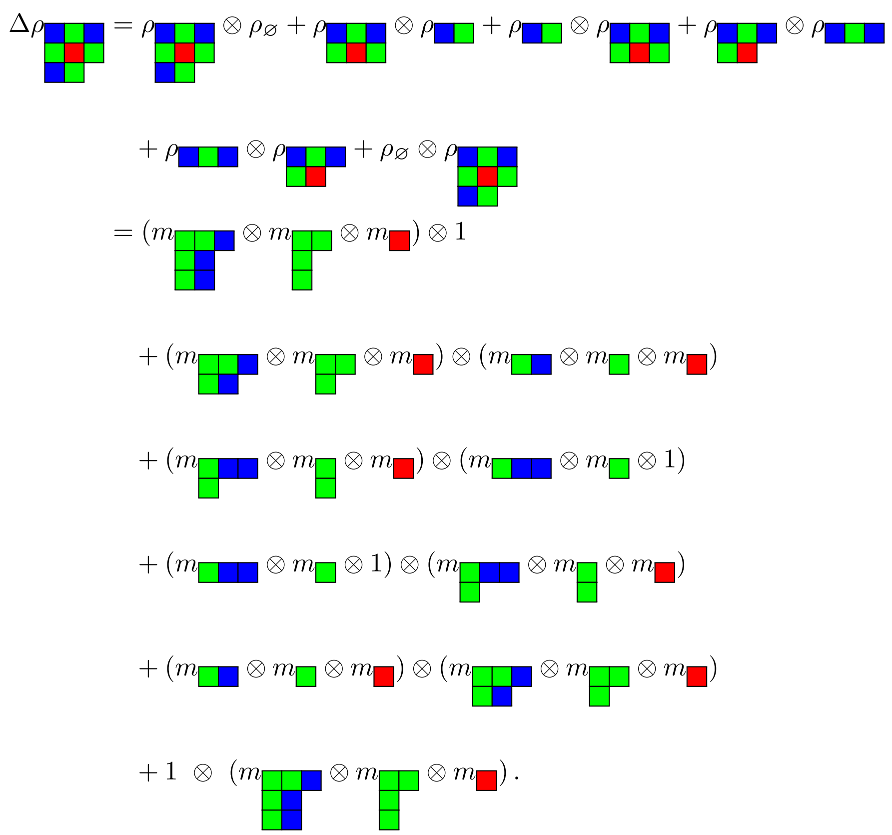
Proposition 5.6. *The primitive basis elements for* Γ *(with respect to the comultiplication* $\Delta^{(e)}$*) are precisely of the form* ρ_λ*, where* $\lambda = (m)$ *for some non-negative integer* m.

Example 5.7.

(1) Let $\lambda = (3,3,2)$. Then we have

$$
\begin{aligned}
\Delta\rho_\lambda =& \Delta\rho_{(3,3,2)}\\
&= \rho_{(3,3,2)}\otimes\rho_{\varnothing}+\rho_{(3,3)}\otimes\rho_{(2)}+\rho_{(2)}\otimes\rho_{(3,3)}+\rho_{(3,2)}\otimes\rho_{(3)}+\rho_{(3)}\otimes\rho_{(3,2)}+\rho_{\varnothing}\otimes\rho_{(3,3,2)}\\
&= \rho_{(3,3,2)}\otimes 1+\rho_{(3,3)}\otimes\rho_{(2)}+\rho_{(2)}\otimes\rho_{(3,3)}+\rho_{(3,2)}\otimes\rho_{(3)}+\rho_{(3)}\otimes\rho_{(3,2)}+1\otimes\rho_{(3,3,2)}\\
&= (m_{(3,2,2)}\otimes m_{(2,1,1)}\otimes m_{(1)})\otimes 1+(m_{(3,2)}\otimes m_{(2,1)}\otimes m_{(1)})\otimes(m_{(2)}\otimes m_{(1)}\otimes 1)\\
&+(m_{(2)}\otimes m_{(1)}\otimes 1)\otimes(m_{(3,2)}\otimes m_{(2,1)}\otimes m_{(1)})+(m_{(3,1)}\otimes m_{(1,1)}\otimes 1)\otimes(m_{(3)}\otimes m_{(1)}\otimes 1)\\
&= (m_{(3,2,2)}\otimes m_{(2,1,1)}\otimes m_{(1)})\otimes 1+(m_{(3,2)}\otimes m_{(2,1)}\otimes m_{(1)})\otimes(m_{(2)}\otimes m_{(1)}\otimes 1)\\
&+(m_{(2)}\otimes m_{(1)}\otimes 1)\otimes(m_{(3,2)}\otimes m_{(2,1)}\otimes m_{(1)})+(m_{(3,1)}\otimes m_{(1,1)}\otimes 1)\otimes(m_{(3)}\otimes m_{(1)}\otimes 1)\\
&+(m_{(3)}\otimes m_{(1)}\otimes 1)\otimes(m_{(3,1)}\otimes m_{(1,1)}\otimes m_{(1)})+1\otimes(m_{(3,2,2)}\otimes m_{(2,1,1)}\otimes m_{(1)}).
\end{aligned}
$$

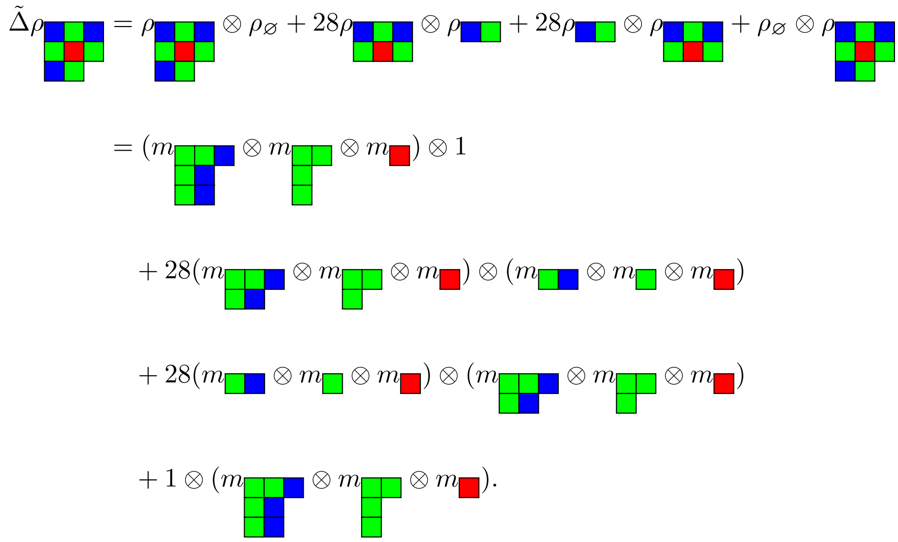
This can be pictured as



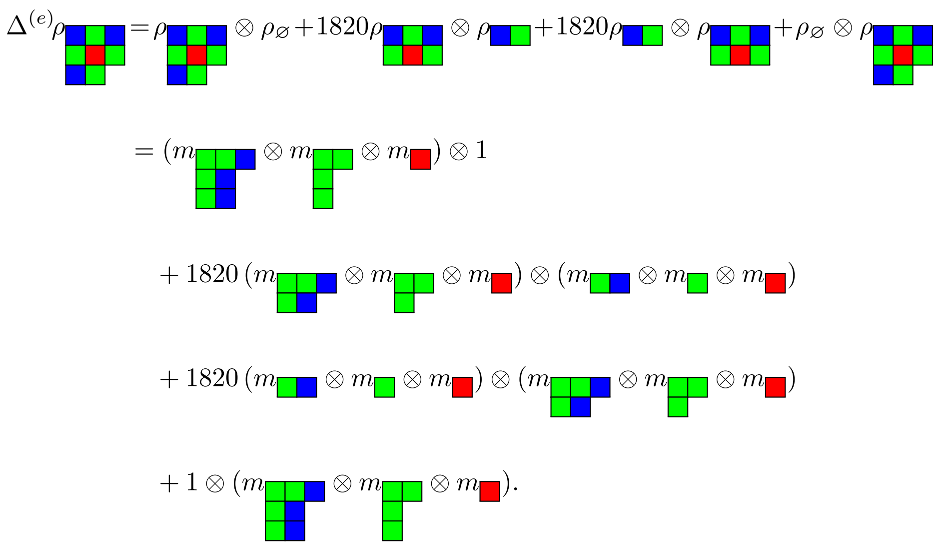
On the other hand, we have

$$\begin{aligned}
\tilde{\Delta}\rho_\lambda &= \tilde{\Delta}\rho_{(3,3,2)}\\
&= \rho_{(3,3,2)} \otimes \rho_{\varnothing} + 28\rho_{(3,3)} \otimes \rho_{(2)} + 28\rho_{(2)} \otimes \rho_{(3,3)} + \rho_{\varnothing} \otimes \rho_{(3,3,2)}\\
&= \rho_{(3,3,2)} \otimes 1 + \rho_{(3,3)} \otimes \rho_{(2)} \quad + \rho_{(2)} \otimes \rho_{(3,3)} + 1 \otimes \rho_{(3,3,2)}\\
&= (m_{(3,2,2)} \otimes m_{(2,1,1)} \otimes m_{(1)}) \otimes 1 + (m_{(3,2)} \otimes m_{(2,1)} \otimes m_{(1)}) \otimes (m_{(2)} \otimes m_{(1)} \otimes m_{(1)})\\
&+ (m_{(2)} \otimes m_{(1)} \otimes m_{(1)}) \otimes (m_{(3,2)} \otimes m_{(2,1)} \otimes m_{(1)}) + 1 \otimes (m_{(3,2,2)} \otimes m_{(2,1,1)} \otimes m_{(1)}).
\end{aligned}$$

This can be visualized as



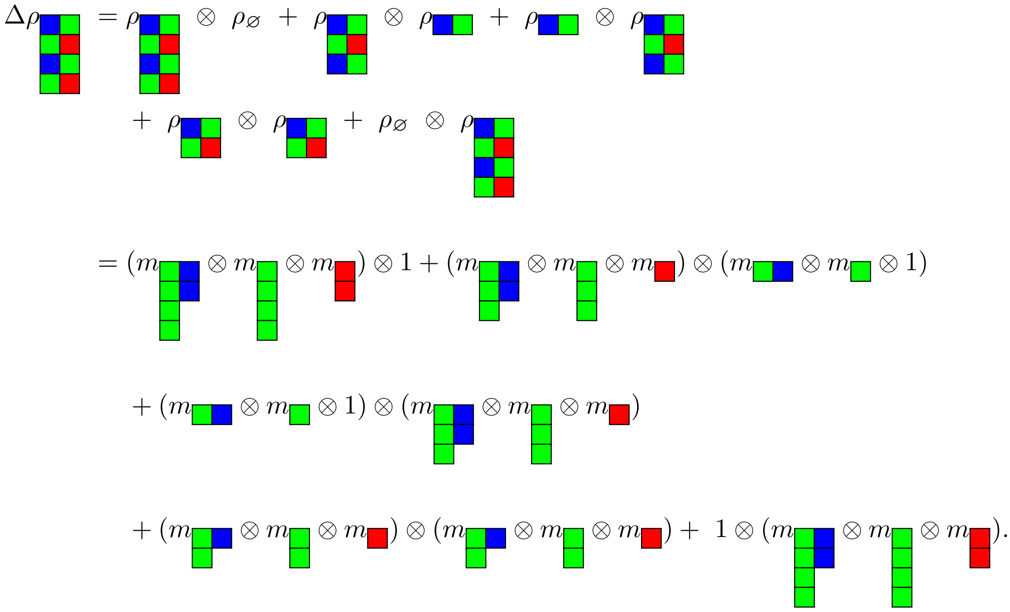
Similarly, one could visualize $\Delta^{(e)}\rho_{(3,3,2)}$ as follows:



(2) To see the difference between Δ, $\tilde{\Delta}$ and $\widehat{\Delta}$ more clearly, let $\lambda = (2,2,2,2)$. Clearly, we have

$$\begin{aligned}
\Delta\rho_\lambda &= \Delta\rho_{(2,2,2,2)}\\
&= \rho_{(2,2,2,2)} \otimes \rho_\varnothing + \rho_{(2,2,2)} \otimes \rho_{(2)} + \rho_{(2)} \otimes \rho_{(2,2,2)} + \rho_{(2,2)} \otimes \rho_{(2,2)} + \rho_\varnothing \otimes \rho_{(2,2,2,2)}\\
&= \rho_{(2,2,2,2)} \otimes 1 + \rho_{(2,2,2)} \otimes \rho_{(2)} + \rho_{(2)} \otimes \rho_{(2,2,2)} + \rho_{(2,2)} \otimes \rho_{(2,2)} + 1 \otimes \rho_{(2,2,2,2)}\\
&= (m_{(2,2,1,1)} \otimes m_{(1,1,1,1)} \otimes m_{(1,1)}) \otimes 1 + (m_{(2,2,1)} \otimes m_{(1,1,1)} \otimes m_{(1)}) \otimes (m_{(2)} \otimes m_{(1)} \otimes 1)\\
&\quad + (m_{(2)} \otimes m_{(1)} \otimes 1) \otimes (m_{(2,2,1)} \otimes m_{(1,1,1)} \otimes m_{(1)})\\
&\quad + (m_{(2,1)} \otimes m_{(1,1)} \otimes m_{(1)}) \otimes (m_{(2,1)} \otimes m_{(1,1)} \otimes m_{(1)})\\
&\quad + 1 \otimes (m_{(2,2,1,1)} \otimes m_{(1,1,1,1)} \otimes m_{(1,1)}),
\end{aligned}$$

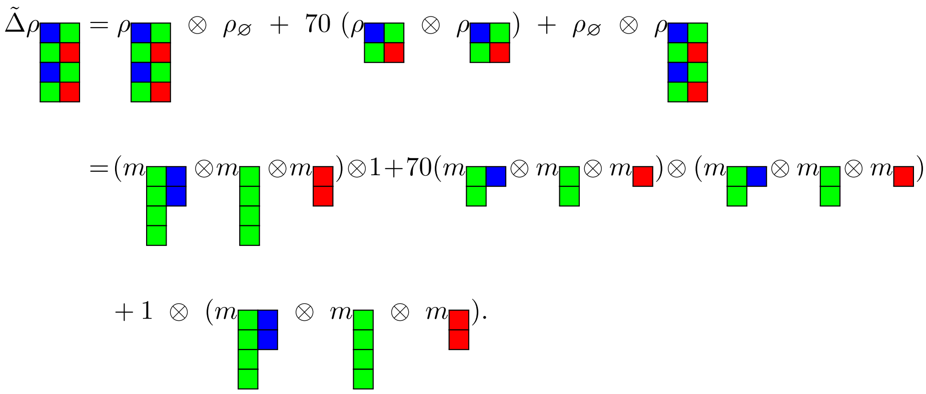
which can be visualized as the following.



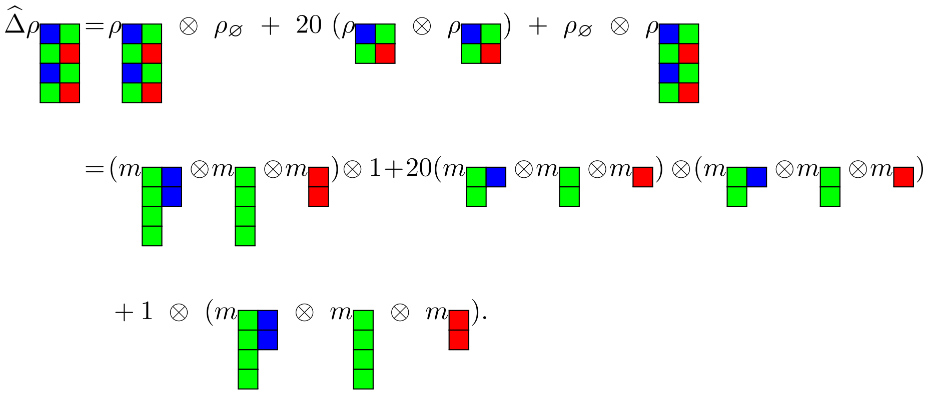
It is easy to check that $\tilde{\Delta}\rho_\lambda$ is given by

$$\begin{aligned}
\tilde{\Delta}\rho_\lambda &= \tilde{\Delta}\rho_{(2,2,2,2)}\\
&= \rho_{(2,2,2,2)} \otimes \rho_\varnothing + \rho_{(2,2)} \otimes \rho_{(2,2)} + \rho_\varnothing \otimes \rho_{(2,2,2,2)}\\
&= \rho_{(2,2,2,2)} \otimes 1 + 70\ (\rho_{(2,2)} \otimes \rho_{(2,2)}) + 1 \otimes \rho_{(2,2,2,2)}\\
&= (m_{(2,2,1,1)} \otimes m_{(1,1,1,1)} \otimes m_{(1,1)}) \otimes 1\\
&+ 70\ (m_{(2,1)} \otimes m_{(1,1)} \otimes m_{(1)}) \otimes (m_{(2,1)} \otimes m_{(1,1)} \otimes m_{(1)})\\
&+ 1 \otimes (m_{(2,2,1,1)} \otimes m_{(1,1,1,1)} \otimes m_{(1,1)})
\end{aligned}$$

One might visualize $\tilde{\Delta}\rho_\lambda$ as follows:



Notably, $\widehat{\Delta}\rho_{(2,2,2,2)}$ looks very similar to $\tilde{\Delta}\rho_{(2,2,2,2)}$. Indeed, the only difference between them is their coefficients. Explicitly, we have



Consider the diagrams:

$$
\begin{tikzcd}
\Gamma\otimes\Gamma \arrow[rr, "\eta"] \arrow[d, "\Delta\otimes\Delta"'] & & \Gamma \arrow[dd, "\Delta"'] \\
\Gamma\otimes\Gamma\otimes\Gamma\otimes\Gamma \arrow[d, "id\otimes\theta\otimes id"'] & & \\
\Gamma\otimes\Gamma\otimes\Gamma\otimes\Gamma \arrow[rr, "\eta\otimes\eta"'] & & \Gamma\otimes\Gamma
\end{tikzcd}
\quad
\begin{tikzcd}
\Gamma\otimes\Gamma \arrow[r, "\epsilon\otimes\epsilon"] \arrow[d, "\eta"'] & \mathbf{k}\otimes\mathbf{k} \arrow[d, "\eta"] \\
\Gamma \arrow[r, "\epsilon"'] & \mathbf{k}
\end{tikzcd}
\quad
\begin{tikzcd}
\mathbf{k} \arrow[r, "u"] \arrow[d, "\Delta"'] & \Gamma \arrow[d, "\Delta"] \\
\mathbf{k}\otimes\mathbf{k} \arrow[r, "u\otimes u"'] & \Gamma\otimes\Gamma
\end{tikzcd}
\quad
\begin{tikzcd}
\mathbf{k} \arrow[rr, "id"] \arrow[dr, "u"'] & & \mathbf{k} \\
& \Gamma \arrow[ur, "\epsilon"'] &
\end{tikzcd}
\tag{5.6}
$$

where $\theta : \Gamma \otimes \Gamma \to \Gamma \otimes \Gamma$ is the twist map. The proof of the following consequence is obvious.

Proposition 5.8.

(1) *The last three diagrams of (5.6) commute while the map η needs not be a* $\mathbf{k}$*-algebra morphism.*

(2) *The last three diagrams of (5.6) are still commutative if one replaces Δ by $\tilde{\Delta}$.*

(3) *Let $CAlg_{\mathbf{k}}$ be the category of commutative* $\mathbf{k}$*-algebras. Then the assignment*

$$\mathscr{G} : CAlg_{\mathbf{k}} \to CAlg_{\mathbf{k}},\ R \mapsto \Gamma(R),\ R \xrightarrow{f} R' \mapsto (\Gamma(R) \xrightarrow{\epsilon_{\Gamma(R)}} R \xrightarrow{f} R' \xrightarrow{u_{\Gamma(R')}} \Gamma(R'))$$

defines a semiendofunctor of $CAlg_{\mathbf{k}}$. Furthermore, we have

$$\mathscr{G}(R \xrightarrow{id_R} R) = (\Gamma(R) \xrightarrow{\epsilon_{\Gamma(R)}} R \xrightarrow{id_R} R \xrightarrow{u_{\Gamma(R)}} \Gamma(R)) = u_{\Gamma(R)}\epsilon_{\Gamma(R)}$$

the convolutional identity element in $End(\Gamma(R))$.

Let $\lambda \in Par$. Write $\lambda^{(GB,0)} = \lambda$, $\lambda^{(GB,1)} = \lambda_{GB}$ and $\lambda^{(GB,2)} = (\lambda_{GB})_{GB} = (\lambda^{(GB,1)})_{GB}$. Inductively, we have $\lambda^{(GB,t)} = (\lambda^{(GB,t-1)})_{GB}$ for any $t \in \mathbb{N}$ with $t \geq 1$. Similarly, one could define $\lambda^{(R,t)}$.

Definition 5.9. Let $\lambda \in Par$.

(1) The GB-order of λ, denoted by $|\lambda|^{GB}$, is the least positive integer t with $\lambda^{(GB,t)} = (\lambda^{(GB,t-1)})_{GB}$. Note that $|\lambda|^{GB} \geq 1$.

(2) Let $t = |\lambda|^{GB}$. Define the sets

$$Par^{(GB)^t} = \{\lambda \in Par : |\lambda|^{GB} \leq t\}$$

and

$$Par^{(GB,n)^t} = \{\lambda \in Par_n : |\lambda|^{GB} \leq t\}.$$

(3) For any $n \in \mathbb{N}$, let $\boxplus(n)$ denote the partition defined by

$$\boxplus(n) = (\underbrace{n, n, \ldots, n}_{n \text{ times}}).$$

Example 5.10.

(1) To find $|\boxplus(4)|^{GB}$ and $|\boxplus(8)|^{GB}$, one might easily calculate

and

Thus, $|\boxplus(4)|^{GB} = 5$ and $|\boxplus(8)|^{GB} = 8$.

$\boxplus(4)^{(GB,0)}$	$\boxplus(4)^{(GB,1)}$	$\boxplus(4)^{(GB,2)}$	$\boxplus(4)^{(GB,3)}$	$\boxplus(4)^{(GB,4)}$
$\boxplus(4)$	$(4,4,2,2)$	$(4,2,2,1)$	$(4,2,1,1)$	$(4,1,1,1)$

$\boxplus(8)^{(GB,0)}$	$\boxplus(8)$
$\boxplus(8)^{(GB,1)}$	$(8,8,8,8,4,4,4,4)$
$\boxplus(8)^{(GB,2)}$	$(8,8,4,4,4,4,2,2)$
$\boxplus(8)^{(GB,3)}$	$(8,4,4,4,2,2,2,1)$
$\boxplus(8)^{(GB,4)}$	$(8,4,2,2,2,2,1,1)$
$\boxplus(8)^{(GB,5)}$	$(8,2,2,2,1,1,1,1)$
$\boxplus(8)^{(GB,6)}$	$(8,2,1,1,1,1,1,1)$
$\boxplus(8)^{(GB,7)}$	$(8,1,1,1,1,1,1,1)$

(2) Similarly, one could check that $|\boxplus(3)|^{GB} = 5$, $|\boxplus(5)|^{GB} = |\boxplus(6)|^{GB} = 8$ and $|\boxplus(10)|^{GB} = 11$.

Remark 5.11. Let $\lambda \in Par$ and $t \in \mathbb{N}$ with $t \geq 2$.

(1) Clearly, $|\lambda|^{GB} \leq |\lambda|$ for $\lambda \in Par$ with $|\lambda| \geq 1$.

(2) If $\lambda \in Par^{(GB)^t}$, then $\lambda_{GB} \in Par^{(GB)^{(t-1)}}$.

(3) If $\lambda \in Par^{(GB)^t}$, then $\left(\lambda^{(GB,t-1)}\right)_R = \emptyset$. In particular, if $\lambda \in Par^{(GB)^2}$, then $\left(\lambda^{(GB,2)}\right)_R = \emptyset$, $\lambda^{(R,2)} = \emptyset$ and $\left(\lambda^{(GB,1)}\right)_R = \emptyset$.

Consider the map

$$\Gamma(\mathbf{k}) \xrightarrow{\Delta^{GB}} \Gamma(\mathbf{k}) \otimes \Gamma(\mathbf{k}) \tag{5.7}$$

defined $\mathbf{k}$-linearly by

$$\Delta^{GB}\rho_\lambda = \begin{cases} 1 \otimes 1 & \text{if } \lambda = \emptyset \\ \rho_{\lambda^{(GB,|\lambda|^{GB}-1)}} \otimes 1 + 1 \otimes \rho_{\lambda^{(GB,|\lambda|^{GB}-1)}} & \text{if } \lambda \neq \emptyset \end{cases}$$

We have the following proposition.

Proposition 5.12. $(\Gamma(\mathbf{k}), \Delta^{GB})$ *is a nonunital* $\mathbf{k}$*-coalgebra.*

Proof. We have to show that the following diagram is satisfied.

$$\begin{tikzcd} & (\Gamma(\mathbf{k}))^{\otimes 3} & \\ (\Gamma(\mathbf{k}))^{\otimes 2} \arrow[ur, "\Delta^{GB}\otimes id"] & & (\Gamma(\mathbf{k}))^{\otimes 2} \arrow[ul, "id\otimes\Delta^{GB}"'] \\ & \Gamma(\mathbf{k}) \arrow[ul, "\Delta^{GB}"] \arrow[ur, "\Delta^{GB}"'] & \end{tikzcd} \tag{5.8}$$

$$\begin{aligned}(\Delta^{GB}\otimes id)\Delta^{GB}\rho_\lambda &= (\Delta^{GB}\otimes id)(\rho_{\lambda^{(GB,|\lambda|^{GB}-1)}}\otimes 1+1\otimes\rho_{\lambda^{(GB,|\lambda|^{GB}-1)}})\\ &= \Delta^{GB}(\rho_{\lambda^{(GB,|\lambda|^{GB}-1)}})\otimes 1+\Delta^{GB}(1)\otimes\rho_{\lambda^{(GB,|\lambda|^{GB}-1)}}\\ &= (\rho_{\lambda^{(GB,|\lambda|^{GB})}}\otimes 1+1\otimes\rho_{\lambda^{(GB,|\lambda|^{GB})}})\otimes 1+1\otimes 1\otimes\rho_{\lambda^{(GB,|\lambda|^{GB}-1)}}\\ &= \rho_{\lambda^{(GB,|\lambda|^{GB})}}\otimes 1\otimes 1+1\otimes\rho_{\lambda^{(GB,|\lambda|^{GB})}}\otimes 1+1\otimes 1\otimes\rho_{\lambda^{(GB,|\lambda|^{GB}-1)}}\end{aligned}$$

$$\begin{aligned}(id\otimes\Delta^{GB})\Delta^{GB}\rho_\lambda &= (id\otimes\Delta^{GB})(\rho_{\lambda^{(GB,|\lambda|^{GB}-1)}}\otimes 1+1\otimes\rho_{\lambda^{(GB,|\lambda|^{GB}-1)}})\\ &= \rho_{\lambda^{(GB,|\lambda|^{GB}-1)}}\otimes\Delta^{GB}(1)+1\otimes\Delta^{GB}(\rho_{\lambda^{(GB,|\lambda|^{GB}-1)}})\\ &= \rho_{\lambda^{(GB,|\lambda|^{GB}-1)}}\otimes(1\otimes 1)+1\otimes(\rho_{\lambda^{(GB,|\lambda|^{GB})}}\otimes 1\otimes 1+1\otimes\rho_{\lambda^{(GB,|\lambda|^{GB})}})\\ &= \rho_{\lambda^{(GB,|\lambda|^{GB})}}\otimes 1\otimes 1+1\otimes\rho_{\lambda^{(GB,|\lambda|^{GB})}}\otimes 1+1\otimes 1\otimes\rho_{\lambda^{(GB,|\lambda|^{GB})}}\end{aligned}$$

For any $\lambda\in Par$, $\lambda^{(GB,|\lambda|^{GB})}=\lambda^{(GB,|\lambda|^{GB}-1)}$ (by the definition of $|\lambda|^{GB}$). It follows that $(\Delta^{GB}\otimes id)\Delta^{GB}=(id\otimes\Delta^{GB})\Delta^{GB}$. Thus, the diagram (5.8) is commutative, and hence $(\Gamma(\mathbf{k}),\Delta^{GB})$ is a nonunital $\mathbf{k}$-coalgebra. $\square$

Definition 5.13. Fix $t\in\mathbb{N}$ with $t\geq 1$. Let $\Gamma^{(GB,n)^t}(\mathbf{k})$ be the free $\mathbf{k}$-module with the basis $\{\rho_\lambda\}_{\lambda\in Par^{(GB,n)^t}}$. Let $\Gamma^{(GB)^t}(\mathbf{k})=\bigoplus_{n\geq 0}\Gamma^{(GB,n)^t}(\mathbf{k})$. Then the set $\{\rho_\lambda\}_{\lambda\in Par^{(GB)^t}}$ forms a basis for $\Gamma^{(GB)^t}(\mathbf{k})$ over $\mathbf{k}$, and $\Gamma^{(GB)^t}(\mathbf{k})$ is called the (GB,t)-*Bayer Noise module* over $\mathbf{k}$.

Now consider the map

$$\Gamma^{(GB)^t}(\mathbf{k})\xrightarrow{\Delta^{(GB)^t}}\Gamma^{(GB)^t}(\mathbf{k})\otimes\Gamma^{(GB)^t}(\mathbf{k}) \tag{5.9}$$

defined $\mathbf{k}$-linearly by

$$\Delta^{(GB)^t}\rho_\lambda=\begin{cases}1\otimes 1 & \text{if } \lambda=\emptyset\\ \rho_{\lambda^{(GB,t-1)}}\otimes 1+1\otimes\rho_{\lambda^{(GB,t-1)}} & \text{if } \lambda\neq\emptyset.\end{cases}$$

We have the following proposition.

Proposition 5.14. *$(\Gamma^{(GB)^t}(\mathbf{k}),\Delta^{(GB)^t})$ is a nonunital $\mathbf{k}$-coalgebra.*

Proof. The proof is very similar to the proof of Proposition (5.12). $\square$

It is well known that the nonunital $\mathbf{k}$-coalgebras $(\Gamma(\mathbf{k}),\Delta^{GB})$ can be extended for a unital $\mathbf{k}$-coalgebra $(\overline{\Gamma(\mathbf{k})},\overline{\Delta^{GB}},\overline{\epsilon^{GB}})$, where $\overline{\Gamma(\mathbf{k})}=\Gamma(\mathbf{k})\oplus\mathbf{k}$, and $\overline{\epsilon^{GB}}:\overline{\Gamma(\mathbf{k})}=\Gamma(\mathbf{k})\oplus\mathbf{k}\to\mathbf{k}$ is the projection map, and $\overline{\Delta^{GB}}$ is the map

$$\overline{\Gamma(\mathbf{k})}\xrightarrow{\overline{\Delta^{GB}}}\overline{\Gamma(\mathbf{k})}\otimes\overline{\Gamma(\mathbf{k})} \tag{5.10}$$

defined by

$$\overline{\Delta^{GB}}(f+a) = \Delta^{GB}(f) + f \otimes 1 + 1 \otimes f + a(1 \otimes 1)$$

for any $f \in \overline{\Gamma(\mathbf{k})}$ and $a \in \mathbf{k}$. Similarly, the nonunital $(\Gamma^{(GB)^t}(\mathbf{k}), \Delta^{(GB)^t})$ can be extended for a unital $\mathbf{k}$-coalgebra $(\overline{\Gamma^{(GB)^t}(\mathbf{k})}, \overline{\Delta^{(GB)^t}}, \overline{\epsilon^{(GB)^t}})$, where $\overline{\Gamma^{(GB)^t}(\mathbf{k})} = \Gamma^{(GB)^t}(\mathbf{k}) \oplus \mathbf{k}$, and $\overline{\epsilon^{(GB)^t}} : \overline{\Gamma^{(GB)^t}(\mathbf{k})} = \Gamma^{(GB)^t}(\mathbf{k}) \oplus \mathbf{k} \to \mathbf{k}$ is the projection map, and $\overline{\Delta^{(GB)^t}}$ is the map

$$\overline{\Gamma^{(GB)^t}(\mathbf{k})} \xrightarrow{\overline{\Delta^{(GB)^t}}} \overline{\Gamma^{(GB)^t}(\mathbf{k})} \otimes \overline{\Gamma^{(GB)^t}(\mathbf{k})} \tag{5.11}$$

defined by

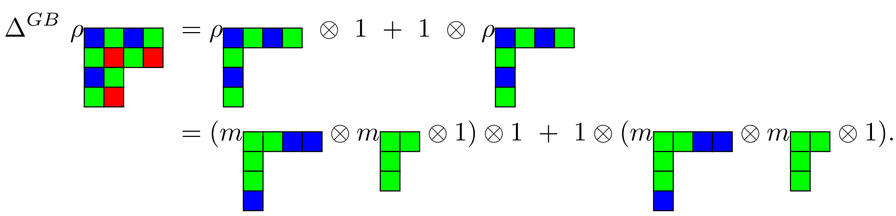
$$\overline{\Delta^{(GB)^t}}(f+a) = \Delta^{(GB)^t}(f) + f \otimes 1 + 1 \otimes f + a(1 \otimes 1)$$

for any $f \in \overline{\Gamma^{(GB)^t}(\mathbf{k})}$ and $a \in \mathbf{k}$. Consequently, we have the following.

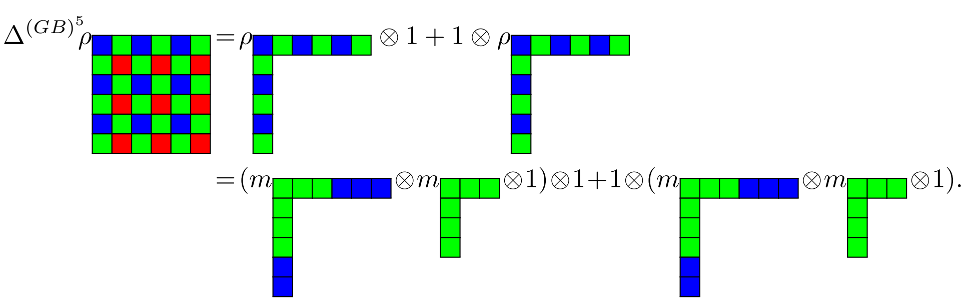
Proposition 5.15. *$(\overline{\Gamma(\mathbf{k})}, \overline{\Delta^{GB}}, \overline{\epsilon^{GB}})$ and $(\overline{\Gamma^{(GB)^t}(\mathbf{k})}, \overline{\Delta^{(GB)^t}}, \overline{\epsilon^{(GB)^t}})$ are (unital) $\mathbf{k}$-coalgebras.*

Example 5.16.

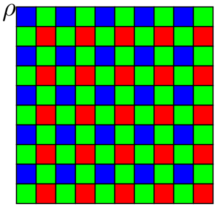
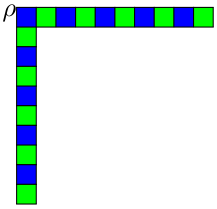
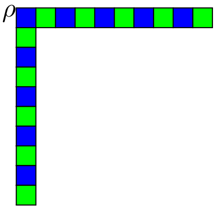
(1) A direct calculation for $\Delta^{GB}\,\rho_{(4,4,2,2)}$ gives the following:

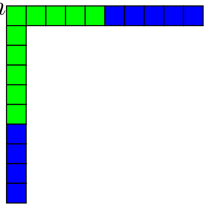
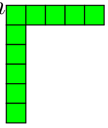


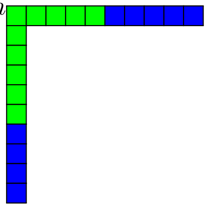
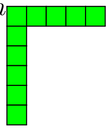
(2) Calculating $\Delta^{(GB)^8}\,\rho_{\boxplus(6)} = \Delta^{(GB)^8}\,\rho_{(6,6,6,6,6,6)}$ gives the following:



(3) One could calculate $\Delta^{(GB)^{11}}\rho_{\boxplus(10)} = \Delta^{(GB)^{11}}\rho_{(10,10,10,10,10,10,10,10,10,10)}$ as follows:

$\Delta^{(GB)^{11}}\rho$  $= \rho$  $\otimes 1 + 1 \otimes \rho$ 

$= (m$  $\otimes\, m$  $\otimes 1) \otimes 1$

$+\ 1 \otimes (m$  $\otimes\, m$  $\otimes 1).$

6. Bayer Noise Functions and Other Bases

Recall that for any $\lambda \in Par$, the *Schur function* is defined to be

$$s_\lambda := \sum_{\mathcal{T}} \mathbf{x}^{cont(\mathcal{T})} \tag{6.1}$$

where $\mathcal{T}$ runs through all semistandard tableaux of shape λ, that is, $\mathcal{T}$ is an assignment of entries in $\{1, 2, 3, \ldots\}$ to the cells of the Young diagram for λ, weakly increasing left-to-right in rows, and strictly increasing top-to-bottom in columns. Here $cont(\mathcal{T})$ denotes the weak composition $\big(|\mathcal{T}^{-1}(1)|, |\mathcal{T}^{-1}(2)|, |\mathcal{T}^{-1}(3)|, \ldots\big)$, so that $\mathbf{x}^{cont(\mathcal{T})} = \prod_i x_i^{|\mathcal{T}^{-1}(i)|}$ [2]. For example,

$$\mathcal{T} = \begin{array}{ccccc} 1 & 1 & 1 & 2 & 7 \\ 2 & 3 & 4 & & \\ 3 & 4 & 4 & & \\ 6 & 7 & & & \end{array}$$

is a semistandard tableaux of shape $\lambda = (5, 3, 3, 2)$ with $\mathbf{x}^{cont(T)} = x_1^3 x_2^2 x_3^2 x_4^3 x_5^0 x_6 x_7^2$. It is well-known that the set $\{s_\lambda\}_{\lambda \in Par}$ forms a $\mathbf{k}$-basis for Λ for any commutative ring $\mathbf{k}$, and for any $\lambda \in Par$, one has

$$s_\lambda = \sum_{\nu \in Par} K_{\lambda,\nu}\ m_\nu,$$

where $K_{\lambda,\nu}$ is the *Kostka number* (a non-negative integer that is equal to the number of semistandard Young tableaux of shape λ and weight ν). This can be used as a inspiration for the following definition.

Definition 6.1. For any $\lambda \in Par$, define the *Bayer Noise Schur function*

$$\delta_\lambda = \sum_{\nu \in Par} K_{\lambda,\nu}\ \rho_\nu \ = \sum_{\nu \in Par} K_{\lambda,\nu}\ m_{\nu_{GB}} \otimes m_{\nu_G} \otimes m_{\nu_R}.$$

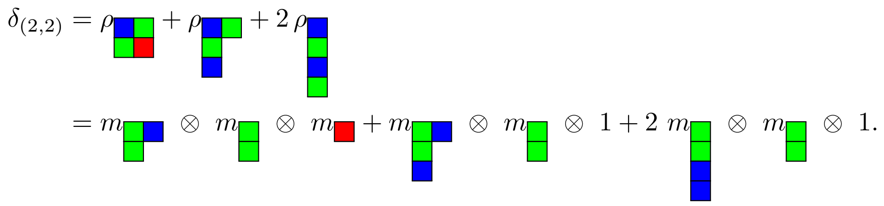
Example 6.2. For $\lambda = (2,2)$, one has

$$\begin{array}{llllllll} s_{(2,2)} & = x_1^2x_2^2 & +x_1^2x_3^2 & +x_1^2x_2x_3 & +x_1^2x_2x_4 & +x_1x_2^2x_3 & +x_1x_2x_3x_4 & +x_1x_2x_3x_4 \ + \cdots \\ & \quad 11 & \quad 11 & \quad 11 & \quad 11 & \quad 12 & \quad 12 & \quad 13 \\ & \quad 22 & \quad 33 & \quad 23 & \quad 24 & \quad 23 & \quad 34 & \quad 24 \\ & = m_{(2,2)} + m_{(2,1,1)} + 2m_{(1,1,1,1)}. \end{array}$$

The Bayer-Schur function $\delta_{(2,2)}$, however, is given by

$$\begin{aligned} \delta_{(2,2)} &= \rho_{(2,2)} + \rho_{(2,1,1)} + 2\rho_{(1,1,1,1)} \\ &= (m_{(2,1)} \otimes m_{(1,1)} \otimes m_{(1)}) + (m_{(2,1,1)} \otimes m_{(1,1)} \otimes m_\emptyset) + 2(m_{(1,1,1,1)} \otimes m_{(1,1)} \otimes m_\emptyset). \end{aligned}$$

This can be visualized as follows:



Recall that the *dominance* or *majorization* order on Par_n is the partial order on the set Par_n whose greater-or-equal relation $\triangleright$ is defined as follows: For two partitions λ and μ of n, we set $\lambda \triangleright \mu$ (and say that λ *dominates*, or *majorizes*, μ) if and only if

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k \geq \mu_1 + \mu_2 + \cdots + \mu_k \quad \text{for } k = 1, 2, \ldots, n \text{ [2]}.$$

It is well-known that the Kostka numbers are triangular with respect to the dominance order. The following is an immediate consequence of the triangularity of the transition matrix of $\{\delta_\lambda\}_{\lambda \in Par}$.

Proposition 6.3. *The set $\{\delta_\lambda\}_{\lambda \in Par}$ forms a $\mathbf{k}$-basis for Γ for any commutative ring $\mathbf{k}$.*

Following [3], define of the families of *power sum symmetric functions* p_n and *elementary symmetric functions* e_n, for $n = 1, 2, 3, \ldots$ by

$$p_n := x_1^n + x_2^n + \cdots = m_{(n)}, \tag{6.2}$$

$$e_n := \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n} = m_{(1^n)}. \tag{6.3}$$

Here,

$$(1^n) = \Bigg(\underbrace{1, 1, \ldots, 1}_{n \text{ ones}}\Bigg).$$

We have the following consequence.

Proposition 6.4. *Let $n \in \mathbb{N}$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in Par$ with $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$.*

(1) We have

$$\rho_{(n)} = \begin{cases} p_n \otimes p_{\frac{n}{2}} \otimes 1 & \textit{if } n \textit{ is even} \\ p_n \otimes p_{\frac{n-1}{2}} \otimes 1 & \textit{if } n \textit{ is odd} \end{cases}$$

(2) We have

$$\rho_{(1^n)} = \begin{cases} e_n \otimes e_{\frac{n}{2}} \otimes 1 & \textit{if } n \textit{ is even} \\ e_n \otimes e_{\frac{n-1}{2}} \otimes 1 & \textit{if } n \textit{ is odd} \end{cases} = \begin{cases} s_{(1^n)} \otimes s_{(1^{\frac{n}{2}})} \otimes 1 & \textit{if } n \textit{ is even} \\ s_{(1^n)} \otimes s_{(1^{\frac{n-1}{2}})} \otimes 1 & \textit{if } n \textit{ is odd} \end{cases}$$

Proof. (1) We have $\rho_{(n)} = m_{(n)_{GB}} \otimes m_{(n)_G} \otimes m_{(n)_R}$. Since $(n)_{GB} = (n)$, $(n)_R = \emptyset$ and

$$(n)_G = \begin{cases} (\frac{n}{2}) & \text{if } n \text{ is even} \\ (\frac{n-1}{2}) & \text{if } n \text{ is odd} \end{cases}$$

Thus,

$$\rho_{(n)} = \begin{cases} p_n \otimes p_{\frac{n}{2}} \otimes 1 & \text{if } n \text{ is even} \\ p_n \otimes p_{\frac{n-1}{2}} \otimes 1 & \text{if } n \text{ is odd} \end{cases}$$

(2) We have $\rho_{(1^n)} = m_{(1^n)_{GB}} \otimes m_{(1^n)_G} \otimes m_{(1^n)_R}$. Since $(1^n)_{GB} = (1^n)$, $(1^n)_R = \emptyset$ and

$$(1^n)_G = \begin{cases} (1^{\frac{n}{2}}) & \text{if } n \text{ is even} \\ (1^{\frac{n-1}{2}}) & \text{if } n \text{ is odd} \end{cases}$$

Thus,

$$\rho_{(1^n)} = \begin{cases} e_n \otimes e_{\frac{n}{2}} \otimes 1 & \text{if } n \text{ is even} \\ e_n \otimes e_{\frac{n-1}{2}} \otimes 1 & \text{if } n \text{ is odd} \end{cases} = \begin{cases} s_{(1^n)} \otimes s_{(1^{\frac{n}{2}})} \otimes 1 & \text{if } n \text{ is even} \\ s_{(1^n)} \otimes s_{(1^{\frac{n-1}{2}})} \otimes 1 & \text{if } n \text{ is odd} \end{cases}$$

$\square$

Recall that a sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ in $\mathbb{N}^\infty$ that have finite support is called a *weak composition*. The nonzero entries of the sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ are called the *parts* of the weak composition α. The sum $\alpha_1 + \alpha_2 + \alpha_3 + \cdots$ of all entries of a weak composition $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$ (or, equivalently, the sum of all parts of α) is called the *size* of α and denoted by $|\alpha|$. A *composition* is a finite tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ of positive integers. In other words, it is a weak composition with no zero entries. We write $\varnothing$ or (0) for the empty composition $()$. Its *length* is defined to be m and denoted by $\ell(\alpha)$; its *size* is defined to be $\alpha_1 + \alpha_2 + \cdots + \alpha_m$ and denoted by $|\alpha|$; its *parts* are its entries $\alpha_1, \alpha_2, \ldots, \alpha_m$. The compositions of size n are called the *compositions of* n. Clearly, any partition of n is a composition of n. Let $Comp_n$ denote the set of all compositions of n, and let $Comp$ denote the set of all compositions. An expansion of a composition α is a weak composition $\bar{\alpha}$ such that removing the zeros from $\bar{\alpha}$ one obtains α. If $\alpha, \beta, \gamma \in Comp$, then we say γ is a shuffle sum of the other two compositions if there are expansions $\bar{\alpha}$ and $\bar{\beta}$ of α and β, respectively, which have length $\ell(\gamma)$ such that $\gamma = \bar{\alpha} + \bar{\beta}$. Here, addition is componentwise [8].

It is well-known that for any $\lambda, \lambda' \in Par$, we have

$$\underline{m}(m_\lambda \otimes m_{\lambda'}) = m_\lambda m_{\lambda'} = \sum_{\substack{\nu \in Par: \\ \nu \vdash |\lambda| + |\lambda'|}} c^\nu_{\lambda,\lambda'} \; m_\nu, \tag{6.4}$$

where $c^\nu_{\lambda,\lambda'}$ is the number of ways of writing ν as a shuffle sum of λ and λ'.

Let $\vartheta : \Gamma^e(\mathbf{k}) \otimes \Gamma^e(\mathbf{k}) \to \Gamma^e(\mathbf{k})$ be a map defined by

$$\begin{aligned} \vartheta(\rho_\lambda \otimes \rho_{\lambda'}) &= \sum_{\substack{\nu \in Par^e : \, \nu_U \vdash |(\lambda + \lambda')_U|, \\ \forall U \in \{GB, R\}}} c^\nu_{\lambda,\lambda'} \; \rho_\nu \\ &= \sum_{\substack{\nu \in Par^e : \, \nu_U \vdash |\lambda_U| + |\lambda'_U|, \\ \forall U \in \{GB, R\}}} c^\nu_{\lambda,\lambda'} \; \rho_\nu \text{ (by 4.3)} , \end{aligned}$$

where $c^\nu_{\lambda,\lambda'}$ is the number of ways of writing ν as a shuffle sum of λ and λ', and the map

$$\mathbf{k} = \Gamma^{(e,0)}(\mathbf{k}) \xrightarrow{u_e} \Gamma^e(\mathbf{k})$$

is the inclusion map.

Let $\delta : \Gamma^e(\mathbf{k}) \to \Gamma^e(\mathbf{k}) \otimes \Gamma^e(\mathbf{k})$ be the map defined $\mathbf{k}$-linearly by

$$\delta\rho_\lambda = \sum_{\substack{(\mu,\nu) \in Par^e \times Par^e : \\ \mu_U \sqcup \nu_U = \lambda_U, \forall U \in \{GB, R\}}} \rho_\mu \otimes \rho_\nu, \tag{6.5}$$

in which $\mu_U \sqcup \nu_U$ is the partition obtained by taking the multiset union of the parts of μ_U and ν_U, and then reordering them to make them weakly decreasing. We end up this paper with the following theorem.

Theorem 6.5.

(1) The triple $(\Gamma^e(\mathbf{k}), \vartheta, u_e)$ *is a* $\mathbf{k}$*-algebra, where*

$$\mathbf{k} = \Gamma^{(e,0)}(\mathbf{k}) \xrightarrow{u_e} \Gamma^e(\mathbf{k})$$

is the inclusion map.

(2) The triple $(\Gamma^e(\mathbf{k}), \delta, \widehat{\epsilon})$ *is a* $\mathbf{k}$*-coalgebra, where* $\Gamma \overset{\widehat{\epsilon}}{\to} \mathbf{k}$ *is the map defined* $\mathbf{k}$*-linearly by*

$$\widehat{\epsilon}|_{\Gamma^{(e,0)}=\mathbf{k}} = id_{\mathbf{k}} \text{ and } \widehat{\epsilon}|_{I=\bigoplus_{n>0}\Gamma^{(e,n)}} = 0.$$

(3) $(\Gamma^e(\mathbf{k}), \vartheta, u_e, \delta, \widehat{\epsilon})$ *is a* $\mathbf{k}$*-bialgebra, and hence* $(\Gamma^e(\mathbf{k}), \vartheta, u_e, \delta, \widehat{\epsilon})$ *is* $\mathbf{k}$*-Hopf algebra.*

Fix a commutative ring $\mathbf{k}$. We end this paper with the following few things as suggestions to the reader who might be interested in.

- Finding a connection between Hall algebras and Bayer Young diagrams.
- Establishing another bases for the Bayer Noise module over $\mathbf{k}$.
- Defining noise symmetric functions using other filters.
- Defining symmetric functions based on the denoising concept.

References

[1] D.R. Bull, *Communicating pictures. A Course in Image and Video Coding,* New York, NY, USA, Academic Press, 2014.

[2] D. Grinberg. V. Reiner. *Hopf Algebras in Combinatorics*, Lecture notes, Vrije Universiteit Brussel, 2020. https://www.cip.ifi.lmu.de/~grinberg/algebra/HopfComb.pdf

[3] I.G. Macdonald, *Symmetric Functions and Hall Polynomials,* 2nd edition, Oxford University Press, Oxford-New York, 1995.

[4] P.L. Méliot, *Representation Theory of Symmetric Groups,* Discrete Mathematics and its Applications, CRC Press, 2017.

[5] A. Mendes, J. Remmel, *Counting with Symmetric Functions*, Developments in Mathematics 43, Springer, 2015.

[6] D. Paliya, A. Foia, R. Bilcub, V. Katkovnika, *Denoising and interpolation of noisy bayer data with adaptive cross-color filters*, In: Proc. of SPIE VCIP (2008).

[7] J.F. Peters, *Topology of Digital Images, Visual Pattern Discovery in Proximity Spaces,* Intelligent Systems Reference Library, 63, Springer, Berlin, 2014.

[8] B.E. Sagan, *Combinatorics: The Art of Counting,* Draft of a textbook, 2020. https://users.math.msu.edu/users/bsagan/Books/Aoc/aoc.pdf.

[9] S.V. Sam, *Notes for Math 740 (Symmetric Functions),* 27 April 2017. https://www.math.wisc.edu/~svs/740/notes.pdf

[10] R.P. Stanley, *Enumerative Combinatorics, Volumes 1 and 2,* Cambridge Studies in Advanced Mathematics 49 and 62, Cambridge University Press, Cambridge, 2nd edition 2011 (volume 1) and 1st edition 1999 (volume 2).

[11] M. Wildon, *An Involutive Introduction to Symmetric Functions,* 1 July 2017. http://www.ma.rhul.ac.uk/~uvah099/teaching.html

DOI: 10.7862/rf.2023.5

Adnan Hashim Abdulwahid
email: AAbdulwahid@tamut.edu, adnanalgebra@gmail.com
ORCID: 0000-0002-4885-7770
College of Business, Engineering, and Technology
Texas A&M University–Texarkana
7101 University Ave
Texarkana, TX, 75503,
USA

Elgaddafi Elamami
email: eelamami@saumag.edu, g.elamami76@gmail.com
ORCID: 0009-0002-3250-3481
University of St. Thomas
3800 Montrose Blvd
Houston, TX 77006
USA

Received 08.08.2022 *Accepted 19.08.2023*