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Asymptotic Analysis for Coupled Parabolic Problem With Dirichlet-Fourier Boundary Conditions in a Thin Domain

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Abstract: This paper concerns the asymptotic behaviour of the initial boundary value problem of a class of reaction-diffusion systems (coupled parabolic systems) posed in a thin domain with Dirichlet-Fourier boundary conditions. We first prove the existence and uniqueness of the solution to the problem for fixed $\varepsilon > 0$ by the Galerkin method. Then, we give the characterization of the limiting behaviour of these solution as the thinness tends to zero.

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1. Introduction

Let Ω^ε be a bounded open subset of $\mathbb{R}^2$ with a sufficiently regular boundary $\partial\Omega^\varepsilon$. We define the thin domain as follows

$$\Omega^\varepsilon = \left\{x = (x_1, x_2) \in \mathbb{R}^2,\ 0 < x_1 < L,\, 0 < x_2 < \varepsilon h\,(x_1)\right\},$$

where $\varepsilon > 0$ is a small parameter that will tend to zero and $h\,(.)$ is a function of class C^1 defined on $[0, L]$ such that

$$0 < \underline{h} = \min_{x_1 \in [0,L]} h(x_1) \leq h(x_1) \leq \overline{h} = \max_{x_1 \in [0,L]} h(x_1),\ \forall x_1 \in [0, L]\,.$$

The boundary of Ω^ε consists of three parts: $\partial\Omega^\varepsilon = \partial\Omega_1^\varepsilon \cup \partial\Omega_2^\varepsilon \cup \partial\Omega_3^\varepsilon$, where $\partial\Omega_1^\varepsilon = \{x \in \partial\Omega^\varepsilon : x_2 = \varepsilon h\,(x_1)\}$ is the upper boundary, $\partial\Omega_3 =]0, L[$ is the bottom boundary and $\partial\Omega_2^\varepsilon = (\{x_1 = 0\} \cup \{x_1 = L\}) \times]0, \varepsilon h(x_1)[$ is the lateral part of the boundary of Ω^ε.

In the thin domain Ω^ε, we are interested in analyzing the behaviour of the solutions, as the parameter $\varepsilon \longrightarrow 0$, of the following coupled parabolic problem with Dirichlet-Fourier boundary conditions

$$\partial_t u^\varepsilon - \mathcal{A}_{\alpha^\varepsilon}(u^\varepsilon) + \lambda^\varepsilon v^\varepsilon = f^\varepsilon \text{ on } \Omega^\varepsilon \times (0,T), \tag{1.1}$$

$$\partial_t v^\varepsilon - \mathcal{A}_{\beta^\varepsilon}(v^\varepsilon) + \lambda^\varepsilon u^\varepsilon = g^\varepsilon \text{ on } \Omega^\varepsilon \times (0,T), \tag{1.2}$$

$$\left.\begin{array}{l} u^\varepsilon = 0 \\ v^\varepsilon = 0 \end{array}\right\} \text{ on } (\partial\Omega_1^\varepsilon \cup \partial\Omega_2^\varepsilon) \times (0,T), \tag{1.3}$$

$$\left.\begin{array}{l} \exists l_1^\varepsilon, r^\varepsilon \in \mathbb{R}_+^* : \partial_{n,\alpha^\varepsilon}(u^\varepsilon) + l_1^\varepsilon u^\varepsilon - r^\varepsilon v^\varepsilon = 0 \\ \exists l_2^\varepsilon, r^\varepsilon \in \mathbb{R}_+^* : \partial_{n,\beta^\varepsilon}(v^\varepsilon) + l_2^\varepsilon v^\varepsilon + r^\varepsilon u^\varepsilon = 0 \end{array}\right\} \text{ on }]0, L[\times (0,T), \tag{1.4}$$

where $\mathcal{A}_{c^\varepsilon}(\cdot)$ is the differential operator given by

$$\mathcal{A}_{c^\varepsilon}(\cdot) = \sum_{i,j=1}^{2} \partial_{x_i}\left[c_{ij}^\varepsilon(x)\,\partial_{x_j}(\cdot)\right],$$

λ^ε is a positive constant, $f^\varepsilon(\cdot)$, $g^\varepsilon(\cdot)$, $c_{ij}^\varepsilon(\cdot)$ are given functions and $\partial_{n,c^\varepsilon}(\cdot) = \sum_{i,j=1}^2 c_{ij}^\varepsilon(x)\,\partial_{x_j}(\cdot).n_j$ indicate the derivative compared to the external normal on the boundary $]0, L[$, such that $n = (0, -1)$ is the unit outward normal to $]0, L[$. We complete the problem $(1.1) - (1.4)$ with the following initial conditions

$$(u^\varepsilon(x,0), v^\varepsilon(x,0)) = (0,0), \quad \forall x \in \Omega^\varepsilon. \tag{1.5}$$

We will deal with the problem $(1.1) - (1.5)$ under the following conditions:

$$c_{ij}^\varepsilon \in L_+^\infty(\Omega^\varepsilon),\ c_{ij}^\varepsilon(\cdot) = c_{ji}^\varepsilon(\cdot),\ 1 \leq i,j \leq 2,$$

also $\exists \mu_c > 0$, such that $\forall \eta \in \mathbb{R}^2$

$$\sum_{i,j=1}^{2} c_{ij}^\varepsilon(x)\,\eta_i\eta_j \geq \mu_c \sum_{i=1}^{2} (\eta_i)^2.$$

The study of thin structures with coarse features, fluids filling fine spheres, or even the process of chemical diffusion in the presence of narrow grains is very common in engineering and applied sciences. Recently, the study of the problems of thin structures has been extended to include many problems arising in applications such as mechanics of solids (thin rods, plates, shells), fluid dynamics (lubrication, meteorological problems, ocean dynamics). We refer to ([17], [11]) for some concrete applied problems.

Analyzing the properties of thin structures and the processes that take place on them and understanding how the micro-geometry of a thinner structure affects the overall properties of a material is a very important issue in engineering and materials design. In this regard, obtaining the specific equations of primitive models allows analysis of how different micro-scales affect primitive problems and allows for study and understanding in more complex situations.

Mathematically, the behaviour of the solutions of partial differential equations dealing with the problems of thin domains is a subject that has been addressed in the literature by different authors, we may mention; In [1], they studied the behaviour of the solutions of nonlinear parabolic problems posed in a domain that degenerates into a line segment (thin domain) which has an oscillating boundary. In the paper [15], the authors investigated the asymptotic behavior of the solutions to the p-Laplacian equation posed in a 2-dimensional open set which degenerates into a line segment when a positive parameter ε goes to zero. In [3], they studied the asymptotic behaviour of the solution of a boundary-value problem for the second-order elliptic equation in the bounded domain $\Omega^\varepsilon \subset \mathbb{R}^3$ with Robin type boundary conditions in the oscillating part of the boundary. The authors in [12], examined the limiting behaviour of dynamics for stochastic reaction-diffusion equations driven by an additive noise and a deterministic non-autonomous forcing on an $(n+1)$-dimensional thin region. A nonuniform Neumann boundary-value problem was considered for the Poisson equation in a thin domain Ω^ε coinciding with two thin rectangles connected through a joint of diameter $O(\varepsilon)$ in [10]. For the Stokes system in a thin domain with slip boundary conditions, we mention the works ([2], [7]). For the case of thin elastic structures, there are many works of literature, we mention for example; The authors in [4], addressed the problem of the junction between 3-dimensional and 2-dimensional linearly elastic structures and various asymptotic developments for the junction between plates. The asymptotic analysis of a dynamical problem of elasticity with non-linear dissipative term and non-linear friction of Tresca type was studied in [5]. Along the same lines, the authors in [6], have proved the asymptotic analysis of the solutions of a linear viscoelastic problem with a dissipative and source terms in a three-dimensional thin domain Ω^ε, with non-linear boundary conditions. The authors in [8], were interested in studying the asymptotic analysis of a mathematical model involving a frictionless contact between an quasi-static electro-viscoelastic and a conductive foundation in a three-dimensional thin domain Ω^ε.

On the other hand, a lot of mathematical systems models have been recently used to study pattern formation in population ecology, morphogenesis, neurobiology, chemical reactor theory, and in other fields, see for example ([18], [16], [9]). These phenomena are usually described by the coupled parabolic systems similar to (1.1)-(1.5).

The main purpose of the paper is to prove the existence and uniqueness of the weak solution for the boundary value problem $(1.1)-(1.5)$, and study the asymptotic behaviour of the solution when ε tends to zero.

The rest of the paper is organized as follows. In Section 2, we derive the weak formulation of the problem and prove the theorem of the existence and uniqueness of the weak solution by the classic Faedo-Galerkin method. In Section 3, we seek to know the behaviour of the solution when the small parameter ε tend to zero. For this purpose, we use the technique of the change of the variable to establish some estimates independent of the parameter ε. These estimates will be useful in order to prove the convergence results and the limit problem.

2. Weak formulation of the problem

For obtain the weak formulation of the problem, we introduce some spaces: let $L^2(\Omega^\varepsilon)$ be the usual Lebesgue space with the norm denoted by $\|\cdot\|_{L^2(\Omega^\varepsilon)}$ and $H^1(\Omega^\varepsilon)$ be the Sobolev space

$$H^1(\Omega^\varepsilon) = \left\{ u \in L^2(\Omega^\varepsilon) : \partial_{x_j} u \in L^2(\Omega^\varepsilon),\ j = 1, 2 \right\}.$$

We denote by $H_0^1(\Omega^\varepsilon)$ the closure of $D(\Omega^\varepsilon)$ in $H^1(\Omega^\varepsilon)$, and $H^{-1}(\Omega^\varepsilon)$ the dual space of $H_0^1(\Omega^\varepsilon)$. Let X a Banach space endowed with the norm $\|\cdot\|_X$, we denotes by $L^2(0, T; X)$ the space of functions $u : (0, T) \to X$ such that $u(t)$ is measurable for dt. This space is a Banach space endowed with the norm

$$\|u\|_{L^2(0.T;X)} = \left(\int_0^T \|u(s)\|_X^2 \, ds \right)^{\frac{1}{2}}.$$

We multiply the equation (1.1) by φ and the equation (1.2) by ψ where $(\varphi, \psi) \in H^1(\Omega^\varepsilon)^2$, then we integrate over Ω^ε and applying Green's formula, we obtain the following weak formulation of the problem

$$\text{Find } (u^\varepsilon, v^\varepsilon) \in (K^\varepsilon)^2 \text{ such that} \tag{2.1}$$

$$(\partial_t u^\varepsilon, \varphi) + a_{\alpha^\varepsilon}(u^\varepsilon, \varphi) + (\lambda^\varepsilon v^\varepsilon, \varphi) + \int_0^L (l_1^\varepsilon u^\varepsilon - r^\varepsilon v) .\varphi dx_1 = (f^\varepsilon, \varphi),\ \forall \varphi \in K^\varepsilon,$$

$$(\partial_t v^\varepsilon, \psi) + a_{\beta^\varepsilon}(v^\varepsilon, \psi) + (\lambda^\varepsilon u^\varepsilon, \psi) + \int_0^L (l_2^\varepsilon v^\varepsilon + r^\varepsilon u^\varepsilon) .\psi dx_1 = (g^\varepsilon, \psi),\ \forall \psi \in K^\varepsilon,$$

$$(u^\varepsilon(x, 0), v^\varepsilon(x, 0)) = (0, 0),$$

where

$$K^\varepsilon = \left\{ \zeta \in H^1(\Omega^\varepsilon) : \zeta = 0 \text{ on } \partial\Omega_1^\varepsilon \cup \partial\Omega_2^\varepsilon \right\},$$

and

$$a_{c^\varepsilon}(\cdot, \cdot) = \sum_{i,j=1}^{2} \int_{\Omega^\varepsilon} c_{ij}^\varepsilon(x)\, \partial_{x_i}(\cdot)\, \partial_{x_j}(\cdot)\, dx.$$

Theorem 1. *Assume that*

$$(f^\varepsilon, g^\varepsilon) \in L^2\left(0, T, L^2(\Omega^\varepsilon)\right)^2.$$

Then, there exists a unique solution $(u^\varepsilon, v^\varepsilon)$ *to problem* (2.1) *such that*

$$(u^\varepsilon, v^\varepsilon) \in L^2\left(0, T, H^1(\Omega^\varepsilon)\right)^2,$$

$$(\partial_t u^\varepsilon, \partial_t v^\varepsilon) \in L^2\left(0, T, L^2(\Omega^\varepsilon)\right)^2.$$

Proof.
A) Uniqueness.

Let $(u_1^\varepsilon, v_1^\varepsilon)$ and $(u_2^\varepsilon, v_2^\varepsilon)$ are two possible solutions. Taking in (2.1) $(\varphi,\psi) = (u_2^\varepsilon - u_1^\varepsilon, v_2^\varepsilon - v_1^\varepsilon)$ (respectively $(\varphi,\psi) = (u_1^\varepsilon - u_2^\varepsilon, v_1^\varepsilon - v_2^\varepsilon)$) in the equation relating to $(u_1^\varepsilon, v_1^\varepsilon)$ (respectively $(u_2^\varepsilon, v_2^\varepsilon)$), we find

$$\begin{aligned}&(\partial_t u_1^\varepsilon, u_2^\varepsilon - u_1^\varepsilon) + a_{\alpha^\varepsilon}(u_1^\varepsilon, u_2^\varepsilon - u_1^\varepsilon) + \lambda^\varepsilon (v_1^\varepsilon, u_2^\varepsilon - u_1^\varepsilon)\\ &+ \int_0^L (l_1^\varepsilon u_1^\varepsilon - r^\varepsilon v_1^\varepsilon)(u_2^\varepsilon - u_1^\varepsilon)\,dx_1\\ &= (f^\varepsilon, u_2^\varepsilon - u_1^\varepsilon),\end{aligned} \tag{2.2}$$

$$\begin{aligned}&(\partial_t u_2^\varepsilon, u_1^\varepsilon - u_2^\varepsilon) + a_{\alpha^\varepsilon}(u_2^\varepsilon, u_1^\varepsilon - u_2^\varepsilon) + \lambda^\varepsilon (v_2^\varepsilon, u_1^\varepsilon - u_2^\varepsilon)\\ &+ \int_0^L (l_1^\varepsilon u_2^\varepsilon - r^\varepsilon v_2)(u_1^\varepsilon - u_2^\varepsilon)\,dx_1\\ &= (f^\varepsilon, u_1^\varepsilon - u_2^\varepsilon),\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}&(\partial_t v_1^\varepsilon, v_2^\varepsilon - v_1^\varepsilon) + a_{\beta^\varepsilon}(v_1^\varepsilon, v_2^\varepsilon - v_1^\varepsilon) + \lambda^\varepsilon (u_1^\varepsilon, v_2^\varepsilon - v_1^\varepsilon)\\ &+ \int_0^L (l_2^\varepsilon v_1^\varepsilon + r u_1)(v_2^\varepsilon - v_1^\varepsilon)\,dx_1\\ &= (g^\varepsilon, v_2^\varepsilon - v_1^\varepsilon),\end{aligned} \tag{2.4}$$

$$\begin{aligned}&(\partial_t v_2^\varepsilon, v_1^\varepsilon - v_2^\varepsilon) + a_{\beta^\varepsilon}(v_2^\varepsilon, v_1^\varepsilon - v_2^\varepsilon) + \lambda^\varepsilon (u_2^\varepsilon, v_1^\varepsilon - v_2^\varepsilon)\\ &+ \int_0^L (l_2^\varepsilon v_2^\varepsilon + r^\varepsilon u_2^\varepsilon)(v_1^\varepsilon - v_2^\varepsilon)\,dx_1\\ &= (g^\varepsilon, v_1^\varepsilon - v_2^\varepsilon),\end{aligned} \tag{2.5}$$

we put $\mathcal{U}^\varepsilon = u_1^\varepsilon - u_2^\varepsilon$, and $\mathcal{V}^\varepsilon = v_1^\varepsilon - v_2^\varepsilon$, thus the sum of (2.2) with (2.3), and (2.4) with (2.5) gives

$$-(\partial_t \mathcal{U}^\varepsilon, \mathcal{U}^\varepsilon) - a_{\alpha^\varepsilon}(\mathcal{U}^\varepsilon, \mathcal{U}^\varepsilon) - \lambda^\varepsilon (\mathcal{V}^\varepsilon, \mathcal{U}^\varepsilon) - l_1^\varepsilon \int_0^L \mathcal{U}^\varepsilon.\mathcal{U}^\varepsilon dx_1 + \int_0^L r^\varepsilon \mathcal{V}^\varepsilon.\mathcal{U}^\varepsilon dx_1 = 0,$$

and

$$-(\partial_t \mathcal{V}^\varepsilon, \mathcal{V}^\varepsilon) - a_{\beta^\varepsilon}(\mathcal{V}^\varepsilon, \mathcal{V}^\varepsilon) - \lambda^\varepsilon (\mathcal{U}^\varepsilon, \mathcal{V}^\varepsilon) - l_2^\varepsilon \int_0^L \mathcal{V}^\varepsilon.\mathcal{V}^\varepsilon dx_1 - \int_0^L r^\varepsilon \mathcal{U}^\varepsilon.\mathcal{V}^\varepsilon dx_1 = 0.$$

Now, adding the two equations above, we find

$$\begin{aligned}&(\partial_t \mathcal{U}^\varepsilon(s), \mathcal{U}^\varepsilon(s)) + a_{\alpha^\varepsilon}(\mathcal{U}^\varepsilon(s), \mathcal{U}^\varepsilon(s)) + a_{\beta^\varepsilon}(\mathcal{V}^\varepsilon(s), \mathcal{V}^\varepsilon(s))\\ &\le -2\lambda^\varepsilon (\mathcal{V}^\varepsilon(s), \mathcal{U}^\varepsilon(s)).\end{aligned} \tag{2.6}$$

On the other hand, we have

$$\int_0^t a_{c^\varepsilon}\left(\Psi\left(s\right),\Psi\left(s\right)\right)ds \geq \mu_c \int_0^t \left\|\Psi\left(s\right)\right\|^2_{H^1(\Omega^\varepsilon)} ds,$$

then, integrating the inequality (2.6) over $(0,t)$, we get

$$\left(\left\|\mathcal{U}^\varepsilon\left(s\right)\right\|^2_{L^2(\Omega^\varepsilon)} + \left\|\mathcal{V}^\varepsilon\left(s\right)\right\|^2_{L^2(\Omega^\varepsilon)}\right) + \int_0^t \left(\mu_\alpha \left\|\mathcal{U}^\varepsilon\left(s\right)\right\|^2_{H^1(\Omega^\varepsilon)} + \mu_\beta \left\|\mathcal{V}^\varepsilon\left(s\right)\right\|^2_{H^1(\Omega^\varepsilon)}\right) ds$$
$$\leq 2\lambda^\varepsilon \int_0^t \left(\left\|\mathcal{U}^\varepsilon\left(s\right)\right\|^2_{L^2(\Omega^\varepsilon)} + \left\|\mathcal{V}^\varepsilon\left(s\right)\right\|^2_{L^2(\Omega^\varepsilon)}\right) ds,$$

now, using Gronwall's lemma, we find

$$\left(\mathcal{U}^\varepsilon\left(s\right),\mathcal{V}^\varepsilon\left(s\right)\right) = (0,0)\,,\ \ \forall s \in (0,T)\,.$$

Thus, we obtain the uniqueness of the solution.

B) Existence.

To show the existence of the solution, we use the Faedo-Galerkin approximation.

Let $\{K^\varepsilon_m\}$ be a family of finite dimensional spaces. It introduces a sequence $\left(w^\varepsilon_j\right)$ of functions having the following properties:

* $w^\varepsilon_j \in K^\varepsilon$, $\forall j = 1,...,m$.
* The family $\{w^\varepsilon_1, w^\varepsilon_2, ..., w^\varepsilon_m\}$ is linearly independent.
* The $K^\varepsilon_m = [w^\varepsilon_1, w^\varepsilon_2, ..., w^\varepsilon_m]$ generated by $\{w^\varepsilon_1, w^\varepsilon_2, ..., w^\varepsilon_m\}$ is dense in K^ε.

Let $(u^\varepsilon_m, v^\varepsilon_m) = \left(u^\varepsilon_m\left(t\right), v^\varepsilon_m\left(t\right)\right)$ be an approximate solution such that

$$u^\varepsilon_m\left(t\right) = \sum_{j=1}^m R_{jm}\left(t\right) w^\varepsilon_j, \quad v^\varepsilon_m\left(t\right) = \sum_{j=1}^m P_{jm}\left(t\right) w^\varepsilon_j,$$

where $R_{jm}\left(t\right)$ and $P_{jm}\left(t\right)$ are determined by the following ordinary differential equations:

$$\left(\partial_t u^\varepsilon_m, w^\varepsilon_j\right) + a_{\alpha^\varepsilon}(u^\varepsilon_m, w^\varepsilon_j) + \lambda^\varepsilon(v^\varepsilon_m, w^\varepsilon_j) + \int_0^L \left(l^\varepsilon_1 u^\varepsilon_m - r^\varepsilon v^\varepsilon_m\right).\left(w^\varepsilon_j\right) dx_1 \tag{2.7}$$
$$= \left(f^\varepsilon, w^\varepsilon_j\right),\ 1 \leq j \leq m,$$
$$\left(\partial_t v^\varepsilon_m, w^\varepsilon_j\right) + a_{\beta^\varepsilon}(v^\varepsilon_m, w^\varepsilon_j) + \lambda^\varepsilon(u^\varepsilon_m, w^\varepsilon_j) + \int_0^L \left(l^\varepsilon_2 v^\varepsilon_m + r^\varepsilon u^\varepsilon_m\right).\left(w^\varepsilon_j\right) dx_1$$
$$= \left(g^\varepsilon, w^\varepsilon_j\right),\ 1 \leq j \leq m,$$

with the initial conditions

$$u_m(x,0) = 0,$$
$$u_m\left(0\right) = \sum_{j=1}^m \gamma_{jm}\left(0\right) w_j \overset{m\to\infty}{\longrightarrow} 0 \text{ in } K^\varepsilon,$$

$$v_m(x,0) = 0,$$
$$v_m\left(0\right) = \sum_{j=1}^m \eta_{jm}\left(0\right) w_j \overset{m\to\infty}{\longrightarrow} 0 \text{ in } K^\varepsilon.$$

Now, we will establish some estimates independent on m.

The first estimate.

By multiplying the first and the second equation of (2.7) by $R_{jm}(t)$ and $P_{jm}(t)$ respectively, then sum over j from 1 to m, we obtain

$$(\partial_t u_m^\varepsilon, u_m^\varepsilon) + a_{\alpha^\varepsilon}(u_m^\varepsilon, u_m^\varepsilon) + \lambda^\varepsilon (v_m^\varepsilon, u_m^\varepsilon) + \int_0^L (l_1^\varepsilon u_m^\varepsilon - r^\varepsilon v_m^\varepsilon) u_m^\varepsilon dx_1 = (f^\varepsilon, u_m^\varepsilon),$$

and

$$(\partial_t v_m^\varepsilon, v_m^\varepsilon) + a_{\beta^\varepsilon}(v_m^\varepsilon, v_m^\varepsilon) + \lambda^\varepsilon (u_m^\varepsilon, v_m^\varepsilon) + \int_0^L (l_2^\varepsilon v_m^\varepsilon + r^\varepsilon u_m^\varepsilon) v_m^\varepsilon dx_1 = (g^\varepsilon, v_m^\varepsilon),$$

by integrating over $(0,t)$ the two equations above, and summing the result, we deduce that

$$\begin{aligned}
&\frac{1}{2}\|u_m^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)}^2 + \frac{1}{2}\|v_m^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)}^2 ds + \mu_\alpha \int_0^t \|u_m^\varepsilon(s)\|_{H^1(\Omega^\varepsilon)}^2 ds \\
&+ \mu_\beta \int_0^t \|v_m^\varepsilon(s)\|_{H^1(\Omega^\varepsilon)}^2 ds + l_1^\varepsilon \int_0^t \int_0^L |u_m^\varepsilon(s)|^2 dx_1 ds + l_2^\varepsilon \int_0^t \int_0^L |v_m^\varepsilon(s)|^2 dx_1 ds \\
&\leq \int_0^t (f^\varepsilon(s), u_m^\varepsilon(s)) ds + \int_0^t (g^\varepsilon(s), v_m(s)) ds - 2\lambda^\varepsilon \int_0^t (v_m^\varepsilon(s), u_m^\varepsilon(s)) ds.
\end{aligned}$$

Now, using the fact that

$$\int_0^t |(f^\varepsilon(s), u_m^\varepsilon(s))| ds \leq \int_0^t \|f^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)}^2 ds + \int_0^t \|u_m^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)}^2 ds,$$

$$\int_0^t |(g^\varepsilon(s), v_m^\varepsilon(s))| ds \leq \int_0^t \|g^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)}^2 ds + \int_0^t \|v_m^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)}^2 ds,$$

and

$$2\lambda^\varepsilon \int_0^t |(v_m^\varepsilon(s), u_m^\varepsilon(s))| ds \leq 2\lambda^\varepsilon \int_0^t \|u_m^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)}^2 ds + 2\lambda^\varepsilon \int_0^t \|v_m^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)}^2 ds,$$

we find the following estimate

$$\|u_m^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} + \|v_m^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} + 2\mu_\alpha \int_0^t \|u_m^\varepsilon(s)\|^2_{H^1(\Omega^\varepsilon)} ds$$

$$+ 2\mu_\beta \int_0^t \|v_m^\varepsilon(s)\|^2_{H^1(\Omega^\varepsilon)} ds + 2l_1^\varepsilon \int_0^t \|u_m^\varepsilon(s)\|^2_{L^2(]0,L[)} ds + 2l_2^\varepsilon \int_0^t \|v_m^\varepsilon(s)\|^2_{L^2(]0,L[)}$$

$$\leq (2 + 4\lambda^\varepsilon) \int_0^t \left(\|u_m^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} + \|v_m^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} \right) ds$$

$$+ 2 \int_0^t \|f^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} ds + 2 \int_0^t \|g^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} ds.$$

After applying Gronwall's lemma in the above inequality, we get

$$\begin{aligned} &\|u_m^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} + \|v_m^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} + \|u_m^\varepsilon(s)\|^2_{L^2(0,T,H^1(\Omega^\varepsilon))} \\ &+ \|v_m^\varepsilon(s)\|^2_{L^2(0,T,H^1(\Omega^\varepsilon))} + \|u_m^\varepsilon(s)\|^2_{L^2(0,T,L^2(]0,L[))} + \|v_m^\varepsilon(s)\|^2_{L^2(0,T,L^2(]0,L[))} \\ &\leq c_T^\varepsilon, \end{aligned} \tag{2.8}$$

where c_T^ε is a constant independent on m.

The second estimate.

We multiply the first and the second equation of (2.7) by $R'_{jm}(\tau)$ and $P'_{jm}(\tau)$ respectively, then sum over j from 1 to m, we have

$$(\partial_t u_m^\varepsilon, \partial_t u_m^\varepsilon) + a_{\alpha^\varepsilon}(u_m^\varepsilon, \partial_t u_m^\varepsilon) + \lambda^\varepsilon (v_m^\varepsilon, \partial_t u_m^\varepsilon) + \int_0^L (l_1^\varepsilon u_m^\varepsilon - r^\varepsilon v_m^\varepsilon) \partial_t u_m^\varepsilon dx_1$$

$$= (f^\varepsilon, \partial_t u_m^\varepsilon),$$

$$(\partial_t v_m^\varepsilon, \partial_t v_m^\varepsilon) + a_{\beta^\varepsilon}(v_m^\varepsilon, \partial_t v_m^\varepsilon) + \lambda^\varepsilon (u_m^\varepsilon, \partial_t v_m^\varepsilon) + \int_0^L (l_2^\varepsilon v_m^\varepsilon + r^\varepsilon u_m^\varepsilon) \partial_t v_m^\varepsilon dx_1$$

$$= (g^\varepsilon, \partial_t v_m^\varepsilon),$$

as $\int_0^t a_{\alpha^\varepsilon}(u_m^\varepsilon(s), \partial_t u_m^\varepsilon(s)) ds$, $\int_0^t \int_0^L (u_m^\varepsilon(s) . \partial_t u_m^\varepsilon(s)) dx_1 ds$, $\int_0^t \int_0^L (v_m^\varepsilon . \partial_t v_m^\varepsilon) dx_1 ds$ and $\int_0^t a_{\beta^\varepsilon}(v_m^\varepsilon(s), \partial_t v_m^\varepsilon(s)) ds$ are positive terms, integrating from 0 to t, we find

$$\int_0^t \|\partial_t u_m^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} ds + \int_0^t \|\partial_t v_m^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} ds + \lambda^\varepsilon \int_0^t (v_m^\varepsilon(s), \partial_t u_m^\varepsilon(s)) ds$$

$$+ \lambda^\varepsilon \int_0^t (u_m^\varepsilon(s), \partial_t v_m^\varepsilon(s))$$

$$\leq \int_0^t (f^\varepsilon(s), \partial_t u_m^\varepsilon(s)) ds + \int_0^t (g^\varepsilon(s), \partial_t v_m^\varepsilon(s))$$

$$- r^\varepsilon \int_0^t \int_0^L u_m^\varepsilon(s) . \partial_t v_m^\varepsilon(s) dx_1 ds + r^\varepsilon \int_0^t \int_0^L v_m^\varepsilon(s) . (\partial_t u_m^\varepsilon(s)) dx_1 ds,$$

Next, by using Cauchy-Schwarz inequality, trace theorem and Young's inequality, we obtain

$$\frac{1}{4}\int_0^t \|\partial_t u_m^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} ds + \frac{1}{4}\int_0^t \|\partial_t v_m^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} ds \tag{2.9}$$
$$\leq 4\lambda^\varepsilon \int_0^t \|v_m^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)} ds + 4\lambda^\varepsilon \int_0^t \|u_m^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)} ds + 4\int_0^t \|f^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)} ds$$
$$+ 4r^\varepsilon C(\Omega^\varepsilon)\int_0^t \left(\|u_m^\varepsilon(s)\|_{L^2(]0,L[)} + \|v_m^\varepsilon(s)\|_{L^2(]0,L[)}\right) ds + 4\int_0^t \|g^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)} ds,$$

on the other hand, by the estimate (2.8), we have

$$\int_0^t \|v_m^\varepsilon(s)\|_{L^2(]0,L[)} ds + \int_0^t \|u_m^\varepsilon(s)\|_{L^2(]0,L[)} ds$$
$$+ \int_0^t \|u_m^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)} ds + \int_0^t \|v_m^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)} ds$$
$$\leq c_T^\varepsilon.$$

So, from (2.9), we deduce that there exists $c_1^\varepsilon > 0$ which does not depend to m such that

$$\|\partial_t u_m^\varepsilon(s)\|^2_{L^2(0,T,L^2(\Omega^\varepsilon))} + \|\partial_t v_m^\varepsilon(s)\|^2_{L^2(0,T,L^2(\Omega^\varepsilon))} \leq c_1^\varepsilon. \tag{2.10}$$

C) Limit procedure.

From (2.8) and (2.10), we conclude that there exists a subsequence of the sequence $(u_m^\varepsilon, v_m^\varepsilon)$, with the same notation, such that

$$\begin{aligned} (u_m^\varepsilon, v_m^\varepsilon) &\rightharpoonup (u^\varepsilon, v^\varepsilon) \text{ weakly in } L^2\left(0,T,H^1(\Omega^\varepsilon)\right)^2, \\ (\partial_t u_m^\varepsilon, \partial_t v_m^\varepsilon) &\rightharpoonup (\partial_t u^\varepsilon, \partial_t v^\varepsilon) \text{ weakly in } L^2\left(0,T,L^2(\Omega^\varepsilon)\right)^2. \end{aligned}$$

Finally, using the arguments in reference [13] and the fact that the space K_m^ε is dense in K^ε, we pass to the limit as m→ 0 in (2.7), we find that u^ε and v^ε satisfy

$$\begin{aligned} &(\partial_t u^\varepsilon, \varphi) + a_{\alpha^\varepsilon}(u^\varepsilon, \varphi) + \lambda^\varepsilon(v^\varepsilon, \varphi) + \int_0^L (l_1^\varepsilon u^\varepsilon - r^\varepsilon v^\varepsilon).(\varphi)\, dx_1 \\ = \;& (f^\varepsilon, \varphi), \forall \varphi \in K^\varepsilon, \end{aligned}$$

and

$$\begin{aligned} &(\partial_t v^\varepsilon, \psi) + a_{\beta^\varepsilon}(v^\varepsilon, \psi) + \lambda^\varepsilon(u^\varepsilon, \psi) + \int_0^L (l_2^\varepsilon v_m^\varepsilon + r^\varepsilon u_m^\varepsilon).(\psi)\, dx_1 \\ = \;& (g^\varepsilon, \psi), \forall \psi \in K^\varepsilon, \end{aligned}$$

this imply that

$$\left.\begin{aligned} \partial_t u^\varepsilon + \mathcal{A}_{\alpha^\varepsilon}(u^\varepsilon) + \lambda^\varepsilon v^\varepsilon = f^\varepsilon, \\ \partial_t v^\varepsilon + \mathcal{A}_{\beta^\varepsilon}(v^\varepsilon) + \lambda^\varepsilon u^\varepsilon = g^\varepsilon, \end{aligned}\right\} \text{ a.e in } \Omega^\varepsilon \times (0,T).$$

Theorem **1** is proved. □

3. Asymptotic analysis of the problem

For the asymptotic analysis of the problem (2.1), we use the approach which consists in transposing the problem initially posed in the domain which depends on a small parameter ε in an equivalent problem posed in the fixed domain which is independent on ε.

3.1. The problem in a fixed domain and some estimates

By introducing the change of variables $z = \frac{x_2}{\varepsilon}$, we get the fixed domain

$$\Omega = \left\{ (x_1, z) \in \mathbb{R}^2, \ 0 < x_1 < L, \, 0 < z < h(x_1) \right\},$$

we denote by $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_3$ its boundary, where $\partial\Omega_1 = \{x \in \partial\Omega : x_2 = h(x_1)\}$, $\partial\Omega_2 = (\{x_1 = 0\} \cup \{x_1 = L\}) \times]0, h(x_1)[$ and $\partial\Omega_3 =]0, L[$.

Now, we define the following functions in Ω

$$u^\varepsilon(x_1, x_2, t) = \hat{u}^\varepsilon(x_1, z, t), \quad v^\varepsilon(x_1, x_2, t) = \hat{v}^\varepsilon(x_1, z, t).$$

Let us assume the following dependence (with respect of ε) of the data

$$\begin{aligned}
\alpha_{ij}^\varepsilon(x_1, x_2) &= \hat{\alpha}_{ij}(x_1, z), \ 1 \le i, j \le 2, \\
\beta_{ij}^\varepsilon(x_1, x_2) &= \hat{\beta}_{ij}(x_1, z), \ 1 \le i, j \le 2, \\
\varepsilon^2 f^\varepsilon(x_1, x_2, t) &= \hat{f}(x_1, z, t), \quad \varepsilon^2 g^\varepsilon(x_1, x_2, t) = \hat{g}(x_1, z, t), \\
\varepsilon^2 \lambda^\varepsilon &= \hat{\lambda}, \ \ \varepsilon l_1^\varepsilon = \hat{l}_1, \quad \varepsilon l_2^\varepsilon = \hat{l}_2, \qquad \varepsilon r^\varepsilon = \hat{r}.
\end{aligned} \tag{3.1}$$

Assuming (3.1), the problem (2.1) leads to the following form

$$\text{Find } (\hat{u}^\varepsilon, \hat{v}^\varepsilon) \in K, \text{ such that} \tag{3.2}$$

$$\int_\Omega \varepsilon^2 \partial_t \hat{u}^\varepsilon \varphi dx_1 dz + \varepsilon^2 \int_\Omega \hat{\alpha}_{11}(x_1, z) (\partial_{x_1} \hat{u}^\varepsilon) (\partial_{x_1} \varphi) dx_1 dz$$

$$+\varepsilon \int_\Omega \hat{\alpha}_{12}(x_1, z) (\partial_{x_1} \hat{u}^\varepsilon) (\partial_z \varphi) dx_1 dz + \varepsilon \int_\Omega \hat{\alpha}_{21}(x_1, z) (\partial_z \hat{u}^\varepsilon) (\partial_{x_1} \varphi) dx_1 dz$$

$$+ \int_\Omega \hat{\alpha}_{22}(x_1, z) (\partial_z \hat{u}^\varepsilon) (\partial_z \varphi) dx_1 dz + \hat{\lambda} \int_\Omega \hat{v}^\varepsilon . \varphi dx_1 dz + \hat{l}_1 \int_0^L \hat{u}^\varepsilon . \varphi dx_1 - \hat{r} \int_0^L \hat{v}^\varepsilon . \varphi dx_1$$

$$= \int_\Omega \hat{f} \varphi dx_1 dz, \quad \forall \varphi \in K,$$

$$\int_{\Omega} \varepsilon^2 \left(\partial_t \hat{v}^\varepsilon\right) \psi dx_1 dz + \varepsilon^2 \int_{\Omega} \hat{\beta}_{11}\left(x_1, z\right)\left(\partial_{x_1} \hat{v}^\varepsilon\right)\left(\partial_{x_1} \psi\right) dx_1 dz \tag{3.3}$$

$$+\varepsilon \int_{\Omega} \hat{\beta}_{12}\left(x_1, z\right)\left(\partial_{x_1} \hat{v}^\varepsilon\right)\left(\partial_z \psi\right) dx_1 dz + \varepsilon \int_{\Omega} \hat{\beta}_{21}\left(x_1, z\right)\left(\partial_z \hat{v}^\varepsilon\right)\left(\partial_{x_1} \psi\right) dx_1 dz$$

$$+\int_{\Omega} \hat{\beta}_{22}\left(x_1, z\right)\left(\partial_z \hat{v}^\varepsilon\right)\left(\partial_z \psi\right) dx_1 dz + \hat{\lambda} \int_{\Omega} \hat{u}^\varepsilon . \varphi dx_1 dz + \hat{l}_2 \int_0^L \hat{v}^\varepsilon . \psi dx_1 + \hat{r} \int_0^L \hat{v}^\varepsilon . \psi dx_1$$

$$= \int_{\Omega} \hat{g} \psi dx_1 dz, \ \forall \psi \in K,$$

$$\left(\hat{u}^\varepsilon(x_1, z, 0), \hat{v}^\varepsilon(x_1, z, 0)\right) = (0, 0)\,, \tag{3.4}$$

where

$$K = \left\{\zeta \in H^1(\Omega) : \zeta = 0 \text{ on } \partial\Omega_1 \cup \partial\Omega_2\right\}.$$

Now, we will obtain estimates on $\hat{u}^\varepsilon$, $\hat{v}^\varepsilon$, $\partial_t \hat{u}^\varepsilon$ and $\partial_t \hat{v}^\varepsilon$. These estimates will be useful in order for obtaining the convergence results and the limit problem.

Theorem 2. *Assume that* f^ε, $g^\varepsilon \in L^2\left(0, T, L^2\left(\Omega^\varepsilon\right)\right)$ *and* $4\hat{\lambda}\bar{h}^2 < \min\left(\mu_\alpha, \mu_\beta\right)$. *Then there exists a constant* c *independent on* ε *such that*

$$\begin{aligned} &\left\|\varepsilon \hat{u}^\varepsilon\right\|^2_{L^2(\Omega)} + \left\|\varepsilon \hat{v}^\varepsilon\right\|^2_{L^2(\Omega)} + \left\|\varepsilon \partial_{x_1} \hat{u}^\varepsilon\right\|^2_{L^2(0,T,L^2(\Omega))} + \left\|\partial_z \hat{u}^\varepsilon\right\|^2_{L^2(0,T,L^2(\Omega))} \\ &+ \left\|\varepsilon \partial_{x_1} \hat{v}^\varepsilon\right\|^2_{L^2(0,T,L^2(\Omega))} + \left\|\partial_z \hat{v}^\varepsilon\right\|^2_{L^2(0,T,L^2(\Omega))} + \left\|u^\varepsilon\left(s\right)\right\|^2_{L^2(0,T,L^2(]0,L[))} \\ &+ \left\|v^\varepsilon\left(s\right)\right\|^2_{L^2(0,T,L^2(]0,L[))} \\ &\leq c, \end{aligned} \tag{3.5}$$

$$\left\|\varepsilon^2 \partial_t \hat{u}^\varepsilon\right\|^2_{L^2(0,T,L^2(\Omega))} + \left\|\varepsilon^2 \partial_t \hat{v}^\varepsilon\right\|^2_{L^2(0,T,L^2(\Omega))} \leq c. \tag{3.6}$$

Proof. Let $(u^\varepsilon, v^\varepsilon)$ be the solution of the problem $(1.1) - (1.2)$. Putting $(\varphi, \psi) = (u^\varepsilon, v^\varepsilon)$ in (2.1), leads to

$$\left(\partial_t u^\varepsilon, u^\varepsilon\right) + a_{\alpha^\varepsilon}\left(u^\varepsilon, u^\varepsilon\right) + \lambda^\varepsilon\left(v^\varepsilon, u^\varepsilon\right) + \int_0^L \left(l_1^\varepsilon u^\varepsilon - r^\varepsilon v^\varepsilon\right) . u^\varepsilon dx_1 = \left(f^\varepsilon, u^\varepsilon\right),$$

and

$$\left(\partial_t v^\varepsilon, v^\varepsilon\right) + a_{\beta^\varepsilon}\left(v^\varepsilon, v^\varepsilon\right) + \lambda^\varepsilon\left(u^\varepsilon, v^\varepsilon\right) + \int_0^L \left(l_2^\varepsilon v^\varepsilon + r^\varepsilon u^\varepsilon\right) . v^\varepsilon dx_1 = \left(g^\varepsilon, v^\varepsilon\right),$$

by integrating the two equalities over $(0,t)$ and summing the result, we get

$$\frac{1}{2}\left(\|u^{\varepsilon}(s)\|^2_{L^2(\Omega^{\varepsilon})}+\|v^{\varepsilon}(s)\|^2_{L^2(\Omega^{\varepsilon})}\right)+\mu_\alpha\int_0^t\|\nabla u^{\varepsilon}(s)\|^2_{L^2(\Omega^{\varepsilon})^2}ds \tag{3.7}$$
$$+\mu_\beta\int_0^t\|\nabla v^{\varepsilon}(s)\|^2_{L^2(\Omega^{\varepsilon})^2}ds+l_1^{\varepsilon}\int_0^t\|u^{\varepsilon}(s)\|^2_{L^2(]0,L[)}ds+l_2^{\varepsilon}\int_0^t\|v^{\varepsilon}(s)\|^2_{L^2(]0,L[)}ds$$
$$\leq\int_0^t\int_{\Omega^{\varepsilon}}|f^{\varepsilon}(s).u^{\varepsilon}(s)|\,dxds+\int_0^t\int_{\Omega^{\varepsilon}}|g^{\varepsilon}(s).v^{\varepsilon}(s)|\,dxds$$
$$+2\lambda^{\varepsilon}\int_0^t\int_{\Omega^{\varepsilon}}|u^{\varepsilon}(s).v^{\varepsilon}(s)|\,dxds.$$

Now, we estimate the right-hand side of the inequality (3.7). Using the Cauchy-Schwarz inequality, Poincaré's inequality

$$\|\varphi\|_{L^2(\Omega^{\varepsilon})}\leq\varepsilon\bar{h}\,\|\nabla\varphi\|_{L^2(\Omega^{\varepsilon})^2},\ \forall\varphi\in K^{\varepsilon},$$

and Young's inequality, we have

$$\int_0^t\int_{\Omega^{\varepsilon}}|f^{\varepsilon}(s).u^{\varepsilon}(s)|\,dxds\ \leq\ \frac{2\varepsilon^2\bar{h}^2}{\mu_\alpha}\int_0^t\|f^{\varepsilon}(s)\|^2_{L^2(\Omega^{\varepsilon})}ds+ \tag{3.8}$$
$$\frac{\mu_\alpha}{2}\int_0^t\|\nabla u^{\varepsilon}(s)\|^2_{L^2(\Omega^{\varepsilon})^2}ds,$$

and

$$\int_0^t\int_{\Omega^{\varepsilon}}|g^{\varepsilon}(s).v^{\varepsilon}(s)|\,dxds\ \leq\ \tfrac{2\varepsilon^2\bar{h}^2}{\mu_\beta}\int_0^t\|g^{\varepsilon}(s)\|^2_{L^2(\Omega^{\varepsilon})}ds+ \tag{3.9}$$
$$\tfrac{\mu_\beta}{2}\int_0^t\|\nabla v^{\varepsilon}(s)\|^2_{L^2(\Omega^{\varepsilon})^2}ds,$$

also, we have

$$\lambda^{\varepsilon}\int_0^t\int_{\Omega^{\varepsilon}}|v^{\varepsilon}(s).u^{\varepsilon}(s)|\,dxds\leq\lambda^{\varepsilon}\int_0^t\|v^{\varepsilon}(s)\|_{L^2(\Omega^{\varepsilon})}.\|u^{\varepsilon}(s)\|_{L^2(\Omega^{\varepsilon})}ds \tag{3.10}$$
$$\leq\hat{\lambda}\bar{h}^2\int_0^t\|\nabla v^{\varepsilon}(s)\|^2_{L^2(\Omega^{\varepsilon})^2}ds+\hat{\lambda}\bar{h}^2\int_0^t\|\nabla u^{\varepsilon}(s)\|^2_{L^2(\Omega^{\varepsilon})^2}ds.$$

Injecting the inequalities (3.8), (3.9) and (3.10) in (3.7), we obtain

$$\begin{aligned} &\|u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} + \|v^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} + \left(\frac{\mu_\alpha}{2} - 2\hat{\lambda}\bar{h}^2\right)\int_0^t \|\nabla u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^2}\, ds \\ &+ \left(\frac{\mu_\beta}{2} - 2\hat{\lambda}\bar{h}^2\right)\int_0^t \|\nabla v^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^2}\, ds + l_1^\varepsilon \int_0^t \|u^\varepsilon(s)\|^2_{L^2(]0,L[)}\, ds \\ &+ l_2^\varepsilon \int_0^t \|v^\varepsilon(s)\|^2_{L^2(]0,L[)}\, ds \\ &\leq \frac{2\varepsilon^2\bar{h}^2}{\mu_\alpha}\int_0^t \|f^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)}\, ds + \frac{2\varepsilon^2\bar{h}^2}{\mu_\beta}\int_0^t \|g^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)}\, ds, \end{aligned} \tag{3.11}$$

as

$$\varepsilon^2 \|f^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} = \varepsilon^{-1}\left\|\hat{f}\right\|^2_{L^2(\Omega)}, \quad \varepsilon^2 \|g^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} = \varepsilon^{-1}\|\hat{g}\|^2_{L^2(\Omega)},$$

we multiply the inequality (3.11) by ε. Then we obtain

$$\begin{aligned} &\varepsilon\|u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} + \varepsilon\|v^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} + \left(\frac{\mu_\alpha}{2} - 2\hat{\lambda}\bar{h}^2\right)\int_0^t \varepsilon\|\nabla u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^2}\, ds \\ &+ \left(\frac{\mu_\beta}{2} - 2\hat{\lambda}\bar{h}^2\right)\int_0^t \varepsilon\|\nabla v^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^2}\, ds + \hat{l}_1 \int_0^t \|u^\varepsilon(s)\|^2_{L^2(]0,L[)}\, ds \\ &+ \hat{l}_2 \int_0^t \|v^\varepsilon(s)\|^2_{L^2(]0,L[)}\, ds \\ &\leq A, \end{aligned} \tag{3.12}$$

where $A = \frac{2\bar{h}^2}{\mu_\alpha}\left\|\hat{f}(t)\right\|^2_{L^2(0,T,L^2(\Omega))} + \frac{2\bar{h}^2}{\mu_\beta}\|\hat{g}(t)\|^2_{L^2(0,T,L^2(\Omega))}$ is a constant independent on ε.

From (3.12), we deduce (3.5).

To show the estimate (3.6), we choose $\hat{\varphi} = \partial_t \hat{u}^\varepsilon$ in (3.2), we find

$$\begin{aligned}
&\int\limits_\Omega \varepsilon^2 \partial_t \hat{u}^\varepsilon \partial_t \hat{u}^\varepsilon dx_1 dz + \varepsilon^2 \int\limits_\Omega \hat{a}_{11}(x_1, z)(\partial_{x_1} \hat{u}^\varepsilon)(\partial_{x_1} \partial_t \hat{u}^\varepsilon) dx_1 dz \\
&+ \varepsilon \int\limits_\Omega \hat{a}_{12}(x_1, z)(\partial_{x_1} \hat{u}^\varepsilon)(\partial_z \partial_t \hat{u}^\varepsilon) dx_1 dz \\
&+ \varepsilon \int\limits_\Omega \hat{a}_{21}(x_1, z)(\partial_z \hat{u}^\varepsilon)(\partial_{x_1} \partial_t \hat{u}^\varepsilon) dx_1 dz \\
&+ \int\limits_\Omega \hat{a}_{22}(x_1, z)(\partial_z \hat{u}^\varepsilon)(\partial_z \partial_t \hat{u}^\varepsilon) dx_1 dz + \hat{\lambda} \int_\Omega \hat{v}^\varepsilon . \partial_t \hat{u}^\varepsilon dx_1 dz \\
&+ \hat{l}_1 \int_0^L \hat{u}^\varepsilon . \varepsilon \partial_t \hat{u}^\varepsilon dx_1 - \hat{r} \int_0^L \hat{v}^\varepsilon . \partial_t \hat{u}^\varepsilon dx_1 \\
&= \int\limits_\Omega \hat{f} . \partial_t \hat{u}^\varepsilon dx_1 dz,
\end{aligned} \tag{3.13}$$

integrating this equalit over $(0, t)$, we deduce that

$$\begin{aligned}
&\int\limits_0^t \|\varepsilon \partial_t \hat{u}^\varepsilon(s)\|^2_{L^2(\Omega)} ds - \hat{\lambda} \int\limits_0^t \int_\Omega \partial_t \hat{v}^\varepsilon(s) . \hat{u}^\varepsilon(s) dx_1 dz ds \\
\leq \ &\int\limits_0^t \int\limits_\Omega \hat{f}(s) . \partial_t \hat{u}^\varepsilon(s) dx_1 dz ds - \hat{\lambda} \int_\Omega \hat{v}^\varepsilon(t) . \hat{u}^\varepsilon(t) dx_1 dz + \hat{r} \int_0^L \hat{v}^\varepsilon(t) . \partial_t \hat{u}^\varepsilon(t) dx_1,
\end{aligned}$$

this leads to

$$\begin{aligned}
&\int\limits_0^t \|\varepsilon \partial_t \hat{u}^\varepsilon(s)\|^2_{L^2(\Omega)} ds - \hat{\lambda} \int\limits_0^t \int_\Omega \partial_t \hat{v}^\varepsilon(s) . \hat{u}^\varepsilon(s) dx_1 dz ds \\
&\leq \frac{1}{\varepsilon} \int\limits_0^t \int\limits_\Omega \hat{f}(s) . (\varepsilon \partial_t \hat{u}^\varepsilon(s)) dx_1 dz ds + \frac{\hat{r}}{\varepsilon} \int_0^L (\hat{v}^\varepsilon(t)) . (\varepsilon \partial_t \hat{u}^\varepsilon(t)) dx_1 \\
&+ \hat{\lambda} \int_\Omega \hat{v}^\varepsilon(t) . \hat{u}^\varepsilon(t) dx_1 dz,
\end{aligned}$$

by using Cauchy-Schwarz inequality, trace theorem and Young's inequality, we obtain

$$\int_0^t \|\varepsilon\partial_t \hat{u}^\varepsilon(s)\|^2_{L^2(\Omega)}\,ds - \hat{\lambda}\int_0^t\int_\Omega \partial_t \hat{v}^\varepsilon(s).\hat{u}^\varepsilon(s)\,dx_1dzds \tag{3.14}$$

$$\leq \frac{4}{\varepsilon^2}\int_0^t \left\|\hat{f}(s)\right\|^2_{L^2(\Omega)} ds + \frac{1}{4}\int_0^t \|\varepsilon\partial_t\hat{u}^\varepsilon(s)\|^2_{L^2(\Omega)^2}\,ds + \hat{\lambda}\,\|\hat{v}^\varepsilon(s)\|^2_{L^2(\Omega)}$$

$$+\,\hat{\lambda}\,\|\hat{u}^\varepsilon(s)\|^2_{L^2(\Omega)} + \frac{4}{\varepsilon^2}\hat{r}^2\left(C_{(\Omega)}\right)^2\int_0^t \|\hat{v}^\varepsilon(s)\|^2_{L^2(]0,L[)}\,ds + \frac{1}{4}\int_0^t \|\varepsilon\partial_t\hat{u}^\varepsilon(s)\|^2_{L^2(\Omega)^2}\,ds.$$

On the other hand, we choose $\psi = \partial_t\hat{v}^\varepsilon$ in (3.3) and we use the same techniques as before that we applied to equality (3.13), we find the following inequality

$$\int_0^t \|\varepsilon\partial_t \hat{v}^\varepsilon(s)\|^2_{L^2(\Omega)}\,ds + \hat{\lambda}\int_0^t\int_\Omega \partial_t \hat{v}^\varepsilon(s).\hat{u}^\varepsilon(s)\,dx_1dzds \tag{3.15}$$

$$\leq \quad \frac{4}{\varepsilon^2}\int_0^t \|\hat{g}(s)\|^2_{L^2(\Omega)}\,ds + \frac{1}{4}\int_0^t \|\varepsilon\partial_t\hat{v}^\varepsilon(s)\|^2_{L^2(\Omega)^2}\,ds$$

$$+\frac{4}{\varepsilon^2}\hat{r}^2\left(C_{(\Omega)}\right)^2\int_0^t \|\hat{u}^\varepsilon(s)\|^2_{L^2(]0;L[)}\,ds + \frac{1}{4}\int_0^t \|\varepsilon\partial_t\hat{v}^\varepsilon(s)\|^2_{L^2(\Omega)^2}\,ds.$$

Now, we add the two inequalities (3.14) and (3.15), then we multiply the result by ε^2. Then we get

$$\int_0^t \left\|\varepsilon^2\partial_t\hat{u}^\varepsilon(s)\right\|^2_{L^2(\Omega)} ds + \int_0^t \left\|\varepsilon^2\partial_t\hat{v}^\varepsilon(s)\right\|^2_{L^2(\Omega)} ds$$

$$\leq 8\int_0^t \left\|\hat{f}(s)\right\|^2_{L^2(\Omega)} ds + 8\int_0^t \|\hat{g}(s)\|^2_{L^2(\Omega)}\,ds + \hat{\lambda}\,\|\varepsilon\hat{v}^\varepsilon\|^2_{L^2(\Omega)} + \hat{\lambda}\,\|\varepsilon\hat{u}^\varepsilon\|^2_{L^2(\Omega)}$$

$$+\,8\hat{r}^2\left(C_{(\Omega)}\right)^2\left(\int_0^t \|\hat{u}^\varepsilon(s)\|^2_{L^2(]0;L[)}\,ds + \int_0^t \|\hat{v}^\varepsilon(s)\|^2_{L^2(]0;L[)}\,ds\right),$$

using the fact that

$$\|\varepsilon\hat{v}^\varepsilon\|^2_{L^2(\Omega)} + \|\varepsilon\hat{u}^\varepsilon\|^2_{L^2(\Omega)} + \|\hat{u}^\varepsilon\|^2_{L^2(0,T,L^2(]0;L[))} + \|\hat{v}^\varepsilon\|^2_{L^2(0,T,L^2(]0;L[))} \leq c,$$

we find, that there is a constant c independent on ε, such that

$$\left\|\varepsilon^2\partial_t\hat{u}^\varepsilon\right\|^2_{L^2(0,T,L^2(\Omega))} + \left\|\varepsilon^2\partial_t\hat{v}^\varepsilon\right\|^2_{L^2(0,T,L^2(\Omega))} \leq c.$$

□

3.2. Study of the limit problem as $\varepsilon \to 0$

In this section we give the system satisfied by the limit of the sequences $(\hat{u}^\varepsilon, \hat{v}^\varepsilon)$ on Ω and the two equations describing the boundary conditions on $]0, L[$, for this purpose we introduce the Banach space

$$V_z = \left\{ \zeta \in L^2(\Omega) : \frac{\partial \zeta}{\partial z} \in L^2(\Omega), \, \zeta = 0 \text{ on } \partial\Omega_1 \right\},$$

with norm

$$\|\zeta\|_{V_z} = \left(\|\zeta\|^2_{L^2(\Omega)} + \left\| \frac{\partial \zeta}{\partial z} \right\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}.$$

We recall that the Poincaré inequality in the fixed domain Ω gives

$$\|\zeta\|_{L^2(\Omega)} \leq \bar{h} \left\| \frac{\partial \zeta}{\partial z} \right\|_{L^2(\Omega)}, \text{ for all } \zeta \in V_z.$$

Theorem 3. *Under the hypotheses of the Theorem 2, there exists* $u^*, v^* \in L^2(0, T; V_z)$ *such that*

$$(\hat{u}^\varepsilon, \hat{v}^\varepsilon) \rightharpoonup (u^*, v^*) \qquad \textit{weakly in } L^2(0, T; V_z)^2, \tag{3.16}$$

$$\begin{aligned} (\varepsilon \partial_{x_1} \hat{u}^\varepsilon, \varepsilon \partial_{x_1} \hat{v}^\varepsilon) &\rightharpoonup (0,0) \quad \textit{weakly in } L^2\left(0, T; L^2(\Omega)\right)^2, \\ (\varepsilon \partial_z \hat{u}^\varepsilon, \varepsilon \partial_z \hat{v}^\varepsilon) &\rightharpoonup (0,0) \quad \textit{weakly in } L^2\left(0, T; L^2(\Omega)\right)^2, \end{aligned} \tag{3.17}$$

$$\left(\varepsilon^2 \partial_t \hat{u}^\varepsilon, \varepsilon^2 \partial_t \hat{v}^\varepsilon\right) \rightharpoonup (0,0) \quad \textit{weakly in } L^2\left(0, T; L^2(\Omega)\right)^2. \tag{3.18}$$

Where (u^*, v^*) *is the weak solution to the limit problem*

$$\left| \begin{array}{l} -\frac{\partial}{\partial z}\left[\hat{\alpha}_{22}(x_1, z) \frac{\partial u^*(x_1,z,t)}{\partial z}\right] + \hat{\lambda} v^*(x_1, z, t) = \hat{f}(x_1, z, t), \\ -\frac{\partial}{\partial z}\left[\hat{\beta}_{22}(x_1, z) \frac{\partial v^*(x_1,z,t)}{\partial z}\right] + \hat{\lambda} u^*(x_1, z, t) = \hat{g}(x_1, z, t), \end{array} \right\} \textit{a.e in } \Omega \times (0, T), \tag{3.19}$$

$$\left| \begin{array}{l} -\hat{\alpha}_{22}(x_1, 0) \partial_z u^*(x_1, 0, t) + \hat{l}_1 u^*(x_1, 0, t) - \hat{r} v^*(x_1, 0, t) = 0, \\ -\hat{\beta}_{22}(x_1, 0) \partial_z v^*(x_1, 0, t) + \hat{l}_1 v^*(x_1, 0, t) + \hat{r} u^*(x_1, 0, t) = 0, \end{array} \right\} \textit{a.e on}]0, L[\times (0, T), \tag{3.20}$$

$$(u^*(x, 0), v^*(x, 0)) = (0, 0).$$

Proof.
By the Theorem 2, there exists a constant c independent of ε such that

$$\int_0^t \|\partial_z \hat{u}^\varepsilon(s)\|^2_{L^2(\Omega)} ds \leq c, \qquad \int_0^t \|\partial_z \hat{v}^\varepsilon(s)\|^2_{L^2(\Omega)} ds \leq c.$$

Using these estimates with the Poincaré inequality in the domain Ω, we get

$$\|\hat{u}^{\varepsilon}(s)\|^2_{L^2(0,T,V_z)} \leq c,$$

and

$$\|\hat{v}^{\varepsilon}(s)\|^2_{L^2(0,T,V_z)} \leq c.$$

So $(\hat{u}^{\varepsilon}, \hat{v}^{\varepsilon})_{\varepsilon}$ is bounded in $L^2(0,T,V_z)^2$, which implies the existence of an element (u^*, v^*) in $L^2(0,T,V_z)^2$ such that $(\hat{u}^{\varepsilon}, \hat{v}^{\varepsilon})_{\varepsilon}$ converges weakly to (u^*, v^*) in $L^2(0,T,V_z)^2$, thus we obtain (3.16). For (3.17) through (3.5) and (3.16). As $(\hat{u}^{\varepsilon}, \hat{v}^{\varepsilon})_{\varepsilon}$ converges weakly to (u^*, v^*) in $L^2(0,T,V_z)^2$ and $\left(\varepsilon^2 \partial_t \hat{u}^{\varepsilon}, \varepsilon^2 \partial_t \hat{v}^{\varepsilon}\right)$ converges weakly to (χ, ζ) in $L^2\left(0,T,L^2(\Omega)\right)^2$, we deduce $(\chi, \zeta) = (0,0)$.

Now, by passage to the limit when ε tends to zero in the variational problem (3.3) – (3.4), and using the convergence results, we deduce

$$\int\limits_{\Omega} \hat{\alpha}_{22}(x_1,z)\,\partial_z u^* \partial_z \varphi dx_1 dz + \hat{\lambda} \int\limits_{\Omega} v^* \varphi dx_1 dz + \int_0^L \left(\hat{l}_1 u^* - \hat{r} v^*\right) \varphi dx_1 \tag{3.21}$$
$$= \int\limits_{\Omega} \hat{f}.\varphi dx_1 dz, \quad \forall \varphi \in K,$$

and

$$\int\limits_{\Omega} \hat{\beta}_{22}(x_1,z)\,\partial_z v^* \partial_z \psi dx_1 dz + \hat{\lambda} \int\limits_{\Omega} u^* \psi dx_1 dz + \int_0^L \left(\hat{l}_2 v^* + \hat{r}\right) \psi dx_1 \tag{3.22}$$
$$= \int\limits_{\Omega} \hat{g}.\psi dx_1 dz, \quad \forall \psi \in K,$$

we choice φ and ψ in $H^1_0(\Omega)$, then using Green's formula, we obtain

$$\begin{aligned} -\int\limits_{\Omega} \partial_z \left[\hat{\alpha}_{22}(x_1,z)\,\partial_z u^*\right] \varphi dx_1 dz + \hat{\lambda} \int\limits_{\Omega} v^* \varphi dx_1 dz &= \int\limits_{\Omega} \hat{f}.\varphi dx_1 dz, \\ -\int\limits_{\Omega} \partial_z \left[\hat{\beta}_{22}(x_1,z)\,\partial_z u^*\right] \psi dx_1 dz + \hat{\lambda} \int\limits_{\Omega} u^* \psi dx_1 dz &= \int\limits_{\Omega} \hat{g}.\psi dx_1 dz, \end{aligned}$$

thus

$$\left.\begin{aligned} -\partial_z \left[\hat{\alpha}_{22}(x_1,z)\,\partial_z u^*(x_1,z,t)\right] + \hat{\lambda} v^*(x_1,z,t) = \hat{f}(x_1,z,t) \\ -\partial_z \left[\hat{\beta}_{22}(x_1,z)\,\partial_z v^*(x_1,z,t)\right] + \hat{\lambda} u^*(x_1,z,t) = \hat{g}(x_1,z,t) \end{aligned}\right\} \text{ in } H^{-1}(\Omega), \tag{3.23}$$

as $\hat{f}, \hat{g} \in L^2\left(0,T;L^2(\Omega)\right)$, then (3.23), is valid a.e in $\Omega \times (0,T)$.

Now, let's go back to the two formulas (3.21) and (3.22), using Green's formula and the fact that $(\varphi,\psi)=(0,0)$ on $\partial\Omega_1\cap\partial\Omega_L$, we deduce

$$\begin{aligned}&\int\limits_{\Omega}\left(-\partial_z\left[\hat{\alpha}_{22}\left(x_1,z\right)\partial_z u^*\right]+\hat{\lambda}v^*-\hat{f}\right)\varphi dx_1dz-\int_0^L\hat{\alpha}_{22}\left(x_1,0\right)\partial_z u^*\varphi dx_1\\&+\hat{l}_1\int_0^L u^*.\varphi dx_1-\hat{r}\int_0^L v^*.\varphi dx_1\\&=0,\quad\forall\varphi\in K,\end{aligned}$$

and

$$\begin{aligned}&\int\limits_{\Omega}\left(-\partial_z\left[\hat{\beta}_{22}\left(x_1,z\right)\partial_z u^*\right]+\hat{\lambda}u-\hat{g}\right)\psi dx_1dz-\int_0^L\hat{b}_{22}\left(x_1,0\right)\partial_z v^*\psi dx_1\\&+\hat{l}_2\int_0^L v^*.\psi dx_1+\hat{r}\int_0^L u^*.\psi dx_1\\&=0,\ \ \forall\psi\in K,\end{aligned}$$

this leads to

$$\left.\begin{array}{l}\int_0^L\left(-\hat{\alpha}_{22}\left(x_1,0\right)\partial_z u^*+\hat{l}_1u^*-\hat{r}v^*\right)\varphi dx_1=0,\\\int_0^L\left(-\hat{\beta}_{22}\left(x_1,0\right)\partial_z v^*+\hat{l}_2v^*-\hat{r}u^*\right)\psi dx_1=0,\end{array}\right\},\ \forall\left(\varphi,\psi\right)\in D\left(]0,L[\right)^2,$$

by the density of $D\left(]0,L[\right)^2$ in $L^2\left(]0,L[\right)^2$, we get (3.20). □

Theorem 4. *Assume that* $\min\left(\min\limits_{(x_1,z)\in\Omega}\left(\hat{\alpha}_{22}\left(x_1,z\right)\right),\min\limits_{(x_1,z)\in\Omega}\left(\hat{\beta}_{22}\left(x_1,z\right)\right)\right)>2\hat{\lambda}$. *Then the weak solution* (u^*,v^*) *of the limit problem is unique and satisfies the following two weak formulas*

$$\begin{aligned}&\int_0^L\Bigg(-\int_0^h\int_0^y\alpha_{22}\left(x_1,\varsigma\right)\partial_\varsigma u^*\left(x_1,\varsigma,t\right)d\varsigma dy+\hat{\lambda}\int_0^h\int_0^y\int_0^\eta v^*\left(x_1,\varsigma,t\right)d\varsigma d\eta dy\\&+\frac{h\left(x_1\right)}{3}\int_0^h\alpha_{22}\left(x_1,\varsigma\right)\partial_\varsigma u^*\left(x_1,\varsigma,t\right)d\varsigma+\tilde{F}\\&-\frac{h\left(x_1\right)}{3}\hat{\lambda}\int_0^h\int_0^\eta v^*\left(x_1,\varsigma,t\right)d\varsigma d\eta\Bigg)\Phi_1'\left(x_1\right)dx_1\\&=0,\ \ \forall\Phi_1\in H^1\left(]0,L[\right),\end{aligned}\tag{3.24}$$

and

$$\int_0^L \left(-\int_0^h \int_0^y \beta_{22}(x_1,\varsigma)\,\partial_\zeta v^*(x_1,\varsigma,t)\,d\varsigma dy + \hat{\lambda}\int_0^h \int_0^y \int_0^\eta u^*(x_1,\varsigma,t)\,d\varsigma d\eta dy \right. \tag{3.25}$$
$$+\frac{h(x_1)}{3}\int_0^h \beta_{22}(x_1,\varsigma)\,\partial_\varsigma v^*(x_1,\varsigma,t)\,d\zeta + \tilde{G}$$
$$\left. -\frac{h(x_1)}{3}\hat{\lambda}\int_0^h \int_0^\eta u^*(x_1,\varsigma,t)\,d\varsigma d\eta \right) \Phi_2'(x_1)\,dx_1$$
$$= 0, \quad \forall \Phi_2 \in H^1(]0,L[),$$

with

$$\begin{aligned} \tilde{F} &= \int_0^h \int_0^y \int_0^\eta \hat{f}(x_1,\varsigma,t)\,d\varsigma d\eta dy - \frac{h(x_1)}{3}\int_0^h \int_0^\eta \hat{f}(x_1,\varsigma,t)\,d\varsigma d\eta, \\ \tilde{G} &= \int_0^h \int_0^y \int_0^\eta \hat{g}(x_1,\varsigma,t)\,d\varsigma d\eta dy - \frac{h(x_1)}{3}\int_0^h \int_0^\eta \hat{g}(x_1,\varsigma,t)\,d\varsigma d\eta. \end{aligned}$$

Proof. To prove the uniqueness result, we suppose that there exist two solutions (u^*, v^*) and (u^{**}, v^{**}) of the variational problem $(3.21) - (3.22)$, we have

$$\int_\Omega \hat{\alpha}_{22}(x_1,z)\,\partial_z u^* \partial_z \varphi dx_1 dz + \hat{\lambda}\int_\Omega v^* \varphi dx_1 dz + \int_0^L \left(\hat{l}_1 u^* - \hat{r} v^*\right).\varphi dx_1 \tag{3.26}$$
$$= \int_\Omega \hat{f}.\varphi dx_1 dz, \forall \varphi \in K,$$

$$\int_\Omega \hat{\alpha}_{22}(x_1,z)\,\partial_z u^{**} \partial_z \varphi dx_1 dz + \hat{\lambda}\int_\Omega v^{**} \varphi dx_1 dz + \int_0^L \left(\hat{l}_1 u^{**} - \hat{r} v^{**}\right).\varphi dx_1 \tag{3.27}$$
$$= \int_\Omega \hat{f}.\varphi dx_1 dz, \forall \varphi \in K,$$

and

$$\int_\Omega \hat{\beta}_{22}(x_1,z)\,\partial_z v^* \partial_z \psi dx_1 dz + \hat{\lambda}\int_\Omega u^* \psi dx_1 dz + \int_0^L \left(\hat{l}_2 v^* + \hat{r} u^*\right).\psi dx_1 \tag{3.28}$$
$$= \int_\Omega \hat{g}.\psi dx_1 dz, \forall \psi \in K,$$

$$\int_{\Omega} \hat{\beta}_{22}(x_1,z)\,\partial_z v^{**}\partial_z\psi dx_1dz + \hat{\lambda}\int_{\Omega} u^{**}\psi dx_1dz + \int_0^L \left(\hat{l}_2 v^{**} + \hat{r}u^{**}\right).\psi dx_1 \tag{3.29}$$
$$= \int_{\Omega} \hat{g}.\psi dx_1dz, \forall\psi \in K.$$

By subtracting the equations (3.26) with (3.27) and (3.28) with (3.29), then we take $\varphi = u^* - u^{**}$ and $\psi = v^* - v^{**}$, we get

$$\int_{\Omega} \hat{\alpha}_{22}(x_1,z)\,|\partial_z u^* - \partial_z u^{**}|^2 dx_1dz + \hat{\lambda}\int_{\Omega}(v^* - v^{**})(u^* - u^{**})\,dx_1dz \tag{3.30}$$
$$+ \hat{l}_1\int_0^L |u^* - u^{**}|^2 dx_1 - \hat{r}\int_0^L (v^* - v^{**}).(u^* - u^{**})\,dx_1$$
$$= 0,$$

and

$$\int_{\Omega} \hat{\beta}_{22}(x_1,z)\,|\partial_z v^* - \partial_z v^{**}|^2 dx_1dz + \hat{\lambda}\int_{\Omega}(u^* - u^{**})(v^* - v^{**})\,dx_1dz \tag{3.31}$$
$$+ \hat{l}_1\int_0^L |v^* - v^{**}|^2 dx_1 + \hat{r}\int_0^L (u^* - u^{**}).(v^* - v^{**})\,dx_1$$
$$= 0.$$

Now, by summing the two equations and applying Young's and Poincare's inequalities, we conclude

$$\left(\min(\hat{\alpha}_{22}) - 2\hat{\lambda}\right)\|u^* - u^{**}\|^2_{L^2(0,T;V_z)} + \left(\min\left(\hat{\beta}_{22}\right) - 2\hat{\lambda}\right)\|v^* - v^{**}\|^2_{L^2(0,T;V_z)} \le 0,$$

then, we obtain

$$(u^*, v^*) = (u^{**}, v^{**}).$$

For prove the two weak formulas, we integrate twice the first and the second equation of (3.19) between 0 and z, we obtain

$$-\int_0^z \alpha_{22}(x_1,\varsigma)\,\partial_\varsigma u^*(x_1,\varsigma,t)\,d\varsigma + \frac{z^2}{2}\alpha_{22}(x_1,0)\,\partial_z u^*(x_1,0,t) \tag{3.32}$$
$$+ \hat{\lambda}\int_0^z\int_0^\eta v^*(x_1,\varsigma,t)\,d\varsigma d\eta$$
$$= \int_0^z\int_0^\eta \hat{f}(x_1,\varsigma,t)\,d\varsigma d\eta,$$

and

$$
\begin{aligned}
&-\int_0^z \beta_{22}(x_1,\varsigma)\,\partial_\varsigma v^*(x_1,\varsigma,t)\,d\zeta + \frac{z^2}{2}\beta_{22}(x_1,0)\,\partial_z v_i^*(x_1,0,t) \\
&+\hat{\lambda}\int_0^z\int_0^\eta u^*(x_1,\varsigma,t)\,d\varsigma d\eta \\
&=\int_0^z\int_0^\eta \hat{g}(x_1,\varsigma,t)\,d\varsigma d\eta,
\end{aligned}
\tag{3.33}
$$

in particular for $z = h(x_1)$, we obtain

$$
\begin{aligned}
&-\int_0^h \alpha_{22}(x_1,\varsigma)\,\partial_\varsigma u^*(x_1,\varsigma,t)\,d\varsigma + \frac{h(x_1)^2}{2}\alpha_{22}(x_1,0)\,\partial_z u_i^*(x_1,0,t) \\
&+\hat{\lambda}\int_0^h\int_0^\eta v^*(x_1,\varsigma,t)\,d\varsigma d\eta \\
&=\int_0^h\int_0^\eta \hat{f}(x_1,\varsigma,t)\,d\varsigma d\eta,
\end{aligned}
$$

and

$$
\begin{aligned}
&-\int_0^h \beta_{22}(x_1,\varsigma)\,\partial_\zeta v^*(x_1,\varsigma,t)\,d\zeta + \frac{h(x_1)^2}{2}\beta_{22}(x_1,0)\,\partial_z v_i^*(x_1,0,t) \\
&+\hat{\lambda}\int_0^h\int_0^\eta u^*(x_1,\varsigma,t)\,d\varsigma d\eta \\
&=\int_0^h\int_0^\eta \hat{g}(x_1,\varsigma,t)\,d\varsigma d\eta.
\end{aligned}
$$

Thus, by integrating (3.32) and (3.33) between 0 and $h(x_1)$, we get (3.24) and (3.25). □

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