Journal of
Mathematics
and Applications

JMA No 47, pp 91-108 (2024)

Pure Strategy Solutions in the Progressive Discrete Silent Duel with Identical Linear Accuracy Functions and Shooting Uniform Jitter

Vadim Romanuke

Abstract: The progressive discrete silent duel is studied modeling limited observability within a system in order to make the best discrete-time decision. The moments to make a decision (to take an action, to shoot a bullet) are scheduled beforehand. The kernel of the duel is skew-symmetric, and the duelists (players) have identical linear accuracy functions. The duel is a finite zero-sum game defined on a subset of the unit square. As the duel starts, time moments of possible shooting become denser by a geometric progression. Apart from the duel beginning and end time moments, every following time moment is the partial sum of the respective geometric series, to which a value of the jitter is added. Regardless of the jitter, both the duelists have the same optimal strategies and the game optimal value is 0 due to the skew-symmetry. The only optimal behavior of the duelist at any positive jitter is to shoot at the positively jittered middle of the duel time span. The only optimal behavior of the duelist in the 3×3 duel at any negative jitter is to shoot at the very end of the duel. In the 4×4 and bigger duels, there is an open interval of the negative jitter, between $\frac{\sqrt{17}-5}{8}$ and 0, at which the duel does not have a pure strategy solution. Value $-\frac{1}{4}$ is the boundary case of the negative jitter, at which the 4×4 duel has four versions of the solution. At any other negative jitter, the only optimal behavior of the duelist in the 4×4 duel is to shoot at the very end of the duel. Bigger duels are more affected by negative jitter. There are two intervals of the pure strategy solution nonexistence in 5×5 and bigger duels, one of which is mentioned above, and the other one approaches to interval $\left(-\frac{1}{2};\ -\frac{1}{4}\right)$ on the left endpoint from the right.

AMS Subject Classification: 91A05, 91A50, 91A55.
Keywords and Phrases: Game of timing; Silent duel; Linear accuracy function; Matrix game; Pure strategy solution; Jittered possible shooting moments.

1. Introduction and motivation

Games of timing that represent a wide class of competitive interaction models are intended to develop rational time decisions for participants under limited system observability [2, 3, 6, 14]. The player (participant) must make a decision of innovation, adoption, response, etc., during a time span on which the game exists [16, 17, 15, 7]. If a decision is made in a two-person game commonly referred to as a duel, the other player either learns it or does not learn it until the duel ends. The latter is the case of the silent duel [19, 7, 9, 1], whose solution heavily depends on whether the game is finite or not [4, 7, 12], apart from the game symmetry [6, 3, 11].

A common pattern of the symmetric silent duel is a zero-sum game

$$\langle X,\ Y,\ K(x,\ y)\rangle \tag{1}$$

defined on unit square

$$X \times Y = [0;\ 1] \times [0;\ 1] \tag{2}$$

with kernel

$$K(x,\ y) = x - y + xy\,\mathrm{sign}\,(y - x)\,, \tag{3}$$

where $X = [0;\ 1]$ and $Y = [0;\ 1]$ are the sets of pure strategies of the duelists, in which the pure strategy is a time moment of possible shooting (i. e., making a decision). Obviously, kernel (3) is skew-symmetric:

$$K(y,\ x) = y - x + yx\,\mathrm{sign}\,(x - y) = -K(x,\ y)\,. \tag{4}$$

Game (1) by (2) and (3) is a silent duel with identical linear accuracy functions of the duelists, which are allowed to shoot at any moment during the duel time span $[0;\ 1]$. Owing to property (4), both the duelists in this duel have the same optimal strategies and the game optimal value is 0 [19, 7, 5].

To get rid of infinite supports in the duelists' optimal strategies [4, 19, 7, 18, 10], a discrete version of duel (1) is considered, where the sets of pure strategies of the duelists are

$$X = \{x_i\}_{i=1}^{N} = Y = \{y_j\}_{j=1}^{N} = T = \{t_q\}_{q=1}^{N} \subset [0;\ 1] \tag{5}$$

by

$$t_q < t_{q+1}\ \ \forall\, q = \overline{1,\ N-1}\ \text{ and }\ t_1 = 0,\ \ t_N = 1\ \text{ for }\ N \in \mathbb{N} \setminus \{1\}\,. \tag{6}$$

The discrete silent duel includes the moments of the duel beginning $x = y = 0$ and duel end $x = y = 1$. By (5), finite symmetric game (1) is a matrix game whose solution is of finite supports only [7, 19]. Moreover, the solution is computed far easier than that in the case of infinite game (1) by $X = [0;\ 1]$ and $Y = [0;\ 1]$.

A specific case of possible shooting moments $\{t_q\}_{q=1}^{N}$ is when they, still obeying (6), are assigned according to a geometrical progression:

$$t_q = \sum_{l=1}^{q-1} 2^{-l} = \frac{2^{q-1}-1}{2^{q-1}} \quad \text{for} \quad q = \overline{2,\ N-1}. \tag{7}$$

In this case, the density of pure strategies of the duelist grows in the geometrical progression as the duelist approaches to the duel end [18, 8]. Apart from the duel beginning and end moments, every following moment is the partial sum of the respective geometric series. However, the precise assignment is not always realizable in practice (e. g., due to finite accuracy in measuring the distance between neighboring moments of possible shooting), so

$$t_q = \xi + \sum_{l=1}^{q-1} 2^{-l} = \xi + \frac{2^{q-1}-1}{2^{q-1}} \quad \text{for} \quad q = \overline{2,\ N-1} \quad \text{and} \quad \xi \in \left(-\frac{1}{2};\ \frac{1}{2^{N-2}}\right) \tag{8}$$

instead of (7). The possible shooting moments $\{t_q\}_{q=2}^{N-1}$ specified by (8) is a shooting uniform jitter, which slightly moves points $\{t_q\}_{q=2}^{N-1}$ by (7) within the duel time span $[0;\ 1]$ not violating their relative order (topology) within $[t_2;\ t_{N-1}]$.

The case of $\xi = 0$ is the known progressive discrete silent duel (PDSD) with identical linear accuracy functions whose solutions are studied in [13]: the pure strategy solution is situation

$$\{x_2,\ y_2\} = \left\{\frac{1}{2},\ \frac{1}{2}\right\} \tag{9}$$

in 3×3 PDSDs and bigger. In PDSDs bigger than the 3×3 PDSD, optimal pure strategy situation (9) is the single one. For a trivial 3×3 PDSD, in which the duelist possesses just one moment of possible shooting between the duel beginning and end moments, any pure strategy situation, not containing the duel beginning moment, is optimal.

2. Objective and tasks to be fulfilled

The objective is to study pure strategy solutions of the PDSD

$$\left\langle \{x_i\}_{i=1}^{N},\ \{y_j\}_{j=1}^{N},\ \mathbf{K}_N \right\rangle \tag{10}$$

with identical linear accuracy functions and shooting uniform jitter by (8), where payoff matrix

$$\mathbf{K}_N = [k_{ij}]_{N\times N} \ \text{ by } \ k_{ij} = K(x_i,\ y_j) = \\ = x_i - y_j + x_i y_j \operatorname{sign}(y_j - x_i) \ \text{ and } \ N \in \mathbb{N}\backslash\{1\} \tag{11}$$

and it is skew-symmetric, i. e. $\mathbf{K}_N = -\mathbf{K}_N^{\mathrm{T}}$ or

$$k_{ij} = -k_{ji} \;\; \forall\, i = \overline{1,\ N} \;\text{ and }\; \forall\, j = \overline{1,\ N}.$$

The primary task is to encompass all existing pure strategy solutions for

$$\xi \in \left(-\frac{1}{2};\ \frac{1}{2^{N-2}} \right) \setminus \{0\}\,. \tag{12}$$

The secondary task is to determine all ξ by (12) such that no pure strategy solution exists. Finally, the solution results are to be summarized along with recapitulating their peculiarities, whereupon the study is discussed and concluded in the last section.

3. Trivial cases

If the duelist is allowed to shoot at either the very beginning or end of the PDSD, this is the most trivial case, where $N = 2$ and the respective payoff matrix (11)

$$\mathbf{K}_2 = [k_{ij}]_{2\times 2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{13}$$

does not depend on ξ. The single optimal solution here is pure strategy situation

$$\{x_2,\ y_2\} = \{1,\ 1\}\,. \tag{14}$$

The case with payoff matrix (13) is not referenced further.

It is worth mentioning that

$$K(x_1,\ y_j) = K(0,\ y_j) = -y_j < 0 \;\; \forall\, j = \overline{2,\ N}$$

and therefore the minimum of the first row of matrix (11) does not exceed $-1 < 0$ and thus the game optimal value $v_{\text{opt}} = 0$ cannot be reached in this row, whichever number N is. So, the first row of matrix (11) cannot be an optimal pure strategy of the first duelist (the first row does not contain saddle points). Due to the skew-symmetry of matrix (11), the stated inference is immediately followed by that the first column does not contain saddle points either (the first column cannot be an optimal pure strategy of the second duelist). In the further consideration, only the inferences on saddle points in definite rows of matrix (11), which imply the same inferences on saddle points in respective columns, will be stated. As only a zero entry of matrix (11) can be a saddle point, then a row containing a negative entry does not contain saddle points; neither does the respective column containing the positive entry. Meanwhile, a nonnegative row contains a saddle point on the main diagonal of the payoff matrix. A row whose entries are positive, except for the main diagonal entry, contains a single saddle point which is the single one in such a duel (all the other $N - 1$ rows of the respective column contain negative entries).

In the next case of triviality, when the shooting, apart from the very beginning and end moments $t_1 = 0$, $t_3 = 1$, is also allowed at moment $t_2 = \frac{1}{2}$, the solution depends on the sign of ξ. The following assertion supplements the abovementioned case of $\xi = 0$ [13].

Theorem 1. *In a PDSD (10) by (8) and (11) for $N = 3$, pure strategy situation*

$$\{x_2,\ y_2\} = \left\{\frac{1}{2} + \xi,\ \frac{1}{2} + \xi\right\} \tag{15}$$

is solely optimal by $\xi > 0$, whereas pure strategy situation

$$\{x_3,\ y_3\} = \{1,\ 1\} \tag{16}$$

is solely optimal by $\xi < 0$.

Proof. Due to $k_{13} = -1$, situation

$$\{x_1,\ y_1\} = \{0,\ 0\} \tag{17}$$

is never optimal in the PDSD. The respective payoff matrix is

$$\mathbf{K}_3 = [k_{ij}]_{3\times 3} = \begin{bmatrix} 0 & -\frac{1}{2} - \xi & -1 \\ \frac{1}{2} + \xi & 0 & 2\xi \\ 1 & -2\xi & 0 \end{bmatrix}. \tag{18}$$

If $\xi > 0$ then the second row of matrix (18) is nonnegative and the third row contains a negative entry. The only zero entry in the second row is k_{22}, whence situation (15) is optimal and it is the single saddle point for the 3×3 PDSD with kernel (18) by $\xi > 0$.

If $\xi < 0$ then the second row of matrix (18) contains a negative entry, and thus the second row does not contain saddle points. The third row is nonnegative and its single zero entry is k_{33}, whence situation (16) is optimal and it is the single saddle point for the 3×3 PDSD with kernel (18) by $\xi < 0$. □

In the further consideration, the case with $\xi > 0$ will be called a positive jitter, and the case with $\xi < 0$ will be called a negative jitter. Time moment

$$t_q = \xi + \frac{2^{q-1} - 1}{2^{q-1}} \ \text{ at } \ q \in \left\{\overline{2,\ N-1}\right\}$$

will be called positively ξ-jittered moment and negatively $|\xi|$-jittered moment by $\xi > 0$ and $\xi < 0$, respectively.

4. The positive jitter duel solution

In fact, Theorem 1 determines the single solution of the 3×3 PDSD with a positive jitter, according to which the best decision is made right after the duel passes its start. The question of whether this property remains for bigger PDSDs is answered by the following assertion.

Theorem 2. *In a PDSD (10) by (8) and (11) for $N \in \mathbb{N}\backslash\{1,\ 2\}$, pure strategy situation (15) is solely optimal by $\xi > 0$.*

Proof. Due to Theorem 1, situation (15) is the single saddle point for $N = 3$. For $N \in \mathbb{N}\backslash\{1,\ 2,\ 3\}$ consider entry k_{22} that is the result of when both the duelists simultaneously shoot at the positively ξ-jittered middle of the duel time span corresponding to situation (15). This entry is in the second row of matrix (11), where

$$K\left(x_2,\ y_1\right) = K\left(\frac{1}{2}+\xi,\ 0\right) = \frac{1}{2}+\xi > 0 \tag{19}$$

and

$$K\left(x_2,\ y_j\right) = K\left(\frac{1}{2}+\xi,\ y_j\right) = \frac{1}{2}+\xi - y_j + \left(\frac{1}{2}+\xi\right)\cdot y_j = \\ = \frac{1}{2}+\xi - \frac{1}{2}y_j + \xi y_j > 0 \ \text{ by } \ j=\overline{3,\ N} \ \text{ and } \ \xi > 0 \tag{20}$$

inasmuch as $\frac{1}{2} - \frac{1}{2}y_j \geqslant 0$ for $0 \leqslant y_j \leqslant 1$. So, the second row of matrix (11), apart from the main diagonal entry k_{22}, is positive and therefore situation (15) is optimal; the second row does not contain any other saddle points. Inequalities (19) and (20) also imply that entries $k_{i2} < 0 \ \forall\, i = \overline{3,\ N}$ in the second column, so saddle point (15) is the single one by $\xi > 0$. □

5. Negative jitter duel solutions

In the 3×3 PDSD with a negative jitter, according to Theorem 1, the best decision is made at the very end of the duel. This rule is generally broken in bigger PDSDs.

Theorem 3. *In a PDSD (10) by (8) and (11) for* $N \in \mathbb{N}\backslash\{1,\ 2,\ 3\}$, *pure strategy situation*

$$\{x_3,\ y_3\} = \left\{\frac{3}{4}+\xi,\ \frac{3}{4}+\xi\right\} \tag{21}$$

is solely optimal by

$$\xi \in \left(-\frac{1}{4};\ \frac{\sqrt{17}-5}{8}\right]. \tag{22}$$

Proof. Inasmuch as

$$K\left(x_2,\ y_N\right) = K\left(\frac{1}{2}+\xi,\ 1\right) = \\ = \frac{1}{2}+\xi - 1 + \frac{1}{2}+\xi = 2\xi < 0 \ \text{ by } \ \xi < 0, \tag{23}$$

the second row of matrix (11) does not contain saddle points, whichever the negative jitter is. In the third row, the first entry is

$$k_{31} = K\left(x_3,\ y_1\right) = K\left(\frac{3}{4}+\xi,\ 0\right) = \frac{3}{4}+\xi > 0 \ \text{ by } \ \xi \in \left(-\frac{1}{2};\ 0\right), \tag{24}$$

the second entry is

$$k_{32} = K\left(x_3,\ y_2\right) = K\left(\frac{3}{4}+\xi,\ \frac{1}{2}+\xi\right) =$$

$$= \frac{3}{4} + \xi - \left(\frac{1}{2} + \xi\right) - \left(\frac{3}{4} + \xi\right)\left(\frac{1}{2} + \xi\right) =$$

$$= -\frac{1}{8} - \frac{5}{4}\xi - \xi^2 \geqslant 0 \text{ by } \xi \in \left[\frac{-\sqrt{17}-5}{8};\ \frac{\sqrt{17}-5}{8}\right], \tag{25}$$

where

$$\frac{-\sqrt{17}-5}{8} < -1 < -\frac{1}{2} < -\frac{1}{4} < \frac{\sqrt{17}-5}{8} < 0. \tag{26}$$

The remaining entries of the third row, apart from (24), (25), $k_{33} = 0$, are

$$k_{3j} = K(x_3,\ y_j) = K\left(\frac{3}{4} + \xi,\ y_j\right) = \frac{3}{4} + \xi - y_j + \left(\frac{3}{4} + \xi\right) \cdot y_j =$$

$$= \frac{3}{4} + \xi - \frac{1}{4}y_j + \xi y_j \ \ \forall\, y_j > \frac{3}{4} + \xi \text{ by } j = \overline{4,\ N}, \tag{27}$$

whence

$$\frac{3}{4} + \xi - \frac{1}{4}y_j + \xi y_j \geqslant \frac{1}{2} + 2\xi > 0 \text{ by } \xi > -\frac{1}{4}. \tag{28}$$

So,

$$k_{3j} = K(x_3,\ y_j) > 0 \ \ \forall\, y_j > \frac{3}{4} + \xi \text{ by } j = \overline{4,\ N} \text{ and } \xi > -\frac{1}{4}. \tag{29}$$

With using (24)—(29), the third row of matrix (11), apart from the main diagonal entry k_{33}, is positive by

$$\xi \in \left(-\frac{1}{4};\ \frac{\sqrt{17}-5}{8}\right). \tag{30}$$

Therefore, situation (21) is solely optimal by (30). If $\xi = \dfrac{\sqrt{17}-5}{8}$ then, according to (25), $k_{32} = 0$, while still entries (27) are positive by (28); but as the second row does not contain saddle points, situation (21) remains solely optimal by (22). □

Inequality (23) means that any negative jitter precludes optimality of situation (15). By a negative jitter, shooting straight after the duel begins (at the time moment following the very beginning) is not optimal. The optimality jumps one moment farther by (22), still being achieved without mixing pure strategies. As it will turn out below, the PDSD is not solved in pure strategies at shallower negative jitter compared to (22).

Theorem 4. *No pure strategy solutions exist in a PDSD (10) by (8) and (11) for* $N \in \mathbb{N} \backslash \{1,\ 2,\ 3\}$ *by*

$$\xi \in \left(\frac{\sqrt{17}-5}{8};\ 0\right). \tag{31}$$

Proof. If (31) holds, then, using (25), $k_{32} < 0$, i.e. the third row does not contain saddle points. As (29) is true (the third row entries above the main diagonal are positive),

$$k_{j3} = -k_{3j} = -K\left(x_3,\ y_j\right) < 0\ \ \forall\, y_j > \frac{3}{4} + \xi$$
$$\text{by}\ \ j = \overline{4,\ N}\ \ \text{and}\ \ \xi > \frac{\sqrt{17}-5}{8} > -\frac{1}{4}, \tag{32}$$

i. e. the third column entries below the main diagonal are negative and rows whose number $i = \overline{4,\ N}$ do not contain saddle points. Consequently, the PDSD is not solved in pure strategies by (31). □

Another interesting aspect is when the negative jitter equals to the left endpoint of the half-open interval in (22). This boundary case is treated differently for the 4×4 PDSD and bigger ones.

Theorem 5. *At* $\xi = -\frac{1}{4}$ *a PDSD (10) by (8) and (11) for* $N \in \mathbb{N}\backslash\{1,\ 2,\ 3,\ 4\}$ *has a single optimal situation*

$$\{x_3,\ y_3\} = \left\{\frac{1}{2},\ \frac{1}{2}\right\}. \tag{33}$$

Proof. If $\xi = -\frac{1}{4}$ then $k_{31} = \frac{1}{2}$, $k_{32} = \frac{1}{8}$,

$$k_{3N} = K\left(x_3,\ y_N\right) = K\left(\frac{1}{2},\ 1\right) = 0 \tag{34}$$

while

$$k_{3j} = K\left(x_3,\ y_j\right) = K\left(\frac{1}{2},\ y_j\right) =$$
$$= \frac{1}{2} - y_j + \frac{1}{2}y_j = \frac{1}{2} - \frac{1}{2}y_j > 0\ \ \forall\, y_j > \frac{3}{4} + \xi = \frac{1}{2}\ \ \text{by}\ \ j = \overline{4,\ N-1}. \tag{35}$$

So, the third row is nonnegative containing a saddle point on the main diagonal, which is situation (33). Besides, inequalities (35) imply that columns whose number $j = \overline{4,\ N-1}$ (or rows whose number $i = \overline{4,\ N-1}$) do not contain saddle points. Despite (34), the N-th column (row) for $N \geqslant 5$ does not contain saddle points either because

$$k_{N4} = K\left(x_N,\ y_4\right) = K\left(1,\ -\frac{1}{4} + \frac{2^3-1}{2^3}\right) =$$
$$= K\left(1,\ \frac{5}{8}\right) = 1 - 2\cdot\frac{5}{8} = -\frac{1}{4} < 0, \tag{36}$$

and thus saddle point (33) is the single one at $\xi = -\frac{1}{4}$ and $N \geqslant 5$. □

Theorem 6. *The* 4×4 *PDSD (10) by (8) and (11) at* $\xi = -\frac{1}{4}$ *has four optimal situations: (33),*

$$\{x_4,\ y_4\} = \{1,\ 1\}, \tag{37}$$

$$\{x_3,\ y_4\} = \left\{\frac{1}{2},\ 1\right\}, \tag{38}$$

$$\{x_4,\ y_3\} = \left\{1,\ \frac{1}{2}\right\}. \tag{39}$$

Proof. It is easy to see that the payoff matrix of the 4×4 PDSD at $\xi = -\frac{1}{4}$ is

$$\mathbf{K}_4 = [k_{ij}]_{4\times 4} = \begin{bmatrix} 0 & -\frac{1}{4} & -\frac{1}{2} & -1 \\ \frac{1}{4} & 0 & -\frac{1}{8} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{8} & 0 & 0 \\ 1 & \frac{1}{2} & 0 & 0 \end{bmatrix}. \tag{40}$$

Payoff matrix (40) has four saddle points (33), (37) — (39). □

Finally, the case when

$$\xi \in \left(-\frac{1}{2};\ -\frac{1}{4}\right) \tag{41}$$

is to be considered. Once again, 4×4 PDSDs are recognized differently from bigger PDSDs, which will be shown in the following two assertions. Besides, a subinterval within interval (41) will be determined, by which the PDSDs bigger than the 4×4 PDSD are not solved in pure strategies.

Theorem 7. *In a PDSD (10) by (8) and (11) for $N \in \mathbb{N} \backslash \{1,\ 2,\ 3,\ 4\}$, pure strategy situation*

$$\{x_N,\ y_N\} = \{1,\ 1\} \tag{42}$$

is solely optimal by

$$\xi \in \left(-\frac{1}{2};\ -\frac{1}{2} + \frac{1}{2^{N-2}}\right]. \tag{43}$$

Proof. Then N-th entry in the third row of matrix (11) is

$$k_{3N} = K(x_3,\ y_N) = K\left(\frac{3}{4} + \xi,\ 1\right) = \frac{1}{2} + 2\xi < 0 \ \text{ by } \ \xi < -\frac{1}{4}, \tag{44}$$

so situation (21) is not optimal. The last row of matrix (11) contains saddle point (42) if

$$k_{Nj} = K(x_N,\ y_j) = K(1,\ y_j) = 1 - 2y_j \geqslant 0 \ \ \forall\, j = \overline{1,\ N} \tag{45}$$

or, briefly,

$$y_j \leqslant \frac{1}{2} \ \ \forall\, j = \overline{2,\ N-1} \tag{46}$$

owing to $y_1 = 0 \leqslant \frac{1}{2}$ and $k_{NN} = 0$ regardless of $y_N = 1 > \frac{1}{2}$. Using (8), inequality (46) is re-written as

$$y_j = \xi + \frac{2^{j-1} - 1}{2^{j-1}} \leqslant \frac{1}{2},$$

whence

$$\xi \leqslant \frac{1}{2^{j-1}} - \frac{1}{2} < -\frac{1}{4} \text{ by } j = \overline{2,\ N-1}. \tag{47}$$

The strict inequality in (47) is $\frac{1}{2^{j-1}} < \frac{1}{4}$ or $2^{j-1} > 4$, which holds $\forall\, j = \overline{4,\ N-1}$ for $N \in \mathbb{N} \setminus \{1,\ 2,\ 3,\ 4\}$ by the least possible value $-\frac{1}{2} + \frac{1}{2^{N-2}}$ of the negative jitter. Therefore, situation (42) is optimal if (43) is true.

If

$$\xi \in \left(-\frac{1}{2};\ -\frac{1}{2} + \frac{1}{2^{N-2}} \right) \text{ for } N \in \mathbb{N} \setminus \{1,\ 2,\ 3,\ 4\} \tag{48}$$

then inequality (46) holds strictly, that is

$$k_{Nj} = K(x_N,\ y_j) = K(1,\ y_j) = 1 - 2y_j > 0 \ \ \forall\, j = \overline{1,\ N-1}, \tag{49}$$

whence the N-th row of matrix (11), apart from the main diagonal entry k_{NN}, is positive and therefore optimal situation (42) is the single one. If

$$\xi = -\frac{1}{2} + \frac{1}{2^{N-2}} \text{ for } N \in \mathbb{N} \setminus \{1,\ 2,\ 3,\ 4\} \tag{50}$$

then

$$y_{N-1} = \xi + \frac{2^{N-2} - 1}{2^{N-2}} = -\frac{1}{2} + \frac{1}{2^{N-2}} + \frac{2^{N-2} - 1}{2^{N-2}} = \frac{1}{2} \tag{51}$$

and

$$y_j < y_{N-1} = \frac{1}{2} \ \ \forall\, j = \overline{2,\ N-2}, \tag{52}$$

where (51) and (52) imply that

$$k_{Nj} = K(x_N,\ y_j) = K(1,\ y_j) = 1 - 2y_j > 0 \ \ \forall\, j = \overline{1,\ N-2}, \tag{53}$$

that is the first $N-2$ entries of the N-th row are positive. Next,

$$k_{N,N-1} = 1 - 2y_{N-1} = 0 = k_{N-1,N},$$

but

$$x_{N-1} = \frac{1}{2},$$

$$y_{N-2} = \xi + \frac{2^{N-3} - 1}{2^{N-3}} = -\frac{1}{2} + \frac{1}{2^{N-2}} + 1 - \frac{1}{2^{N-3}} = \frac{1}{2} - \frac{1}{2^{N-2}},$$

and

$$k_{N-1,N-2} = K\left(x_{N-1},\ y_{N-2}\right) = K\left(\frac{1}{2},\ \frac{1}{2} - \frac{1}{2^{N-2}}\right) =$$
$$= \frac{1}{2} - \frac{1}{2} + \frac{1}{2^{N-2}} - \frac{1}{2} \cdot \left(\frac{1}{2} - \frac{1}{2^{N-2}}\right) =$$
$$= \frac{1}{2^{N-2}} - \frac{1}{4} + \frac{1}{2^{N-1}} = \frac{3}{2^{N-1}} - \frac{1}{4} < 0 \text{ for } N \in \mathbb{N}\backslash\left\{1,\ 2,\ 3,\ 4\right\},$$

which implies that the $(N-1)$-th row of matrix (11) does not contain saddle points by (50). □

Theorem 8. *In the* 4×4 *PDSD (10) by (8) and (11), situation (42) is solely optimal by (41).*

Proof. At $N = 4$

$$\left(-\frac{1}{2};\ -\frac{1}{2} + \frac{1}{2^{N-2}}\right) = \left(-\frac{1}{2};\ -\frac{1}{4}\right).$$

In the respective 4×4 PDSD inequality (46) holds as

$$y_{N-1} = y_3 = \xi + \frac{3}{4} < \frac{1}{2} \text{ by } \xi < -\frac{1}{4}, \tag{54}$$

so situation (42) is optimal as well. Besides, it is solely optimal due to inequality (49) holds after (54). □

Theorem 9. *No pure strategy solutions exist in a PDSD (10) by (8) and (11) for* $N \in \mathbb{N}\backslash\left\{1,\ 2,\ 3,\ 4\right\}$ *by*

$$\xi \in \left(-\frac{1}{2} + \frac{1}{2^{N-2}};\ -\frac{1}{4}\right). \tag{55}$$

Proof. The first row of matrix (11) does not contain saddle points; the second row does not contain saddle points due to (23) holds; the third row does not contain saddle points due to (44) holds. Consider entry k_{nn} in matrix (11) for $n \in \left\{\overline{4,\ N-1}\right\}$ and $N \in \mathbb{N}\backslash\left\{1,\ 2,\ 3,\ 4\right\}$. This entry is the result of when both the duelists shoot at moment

$$t_n = \xi + \frac{2^{n-1} - 1}{2^{n-1}} \tag{56}$$

corresponding to situation

$$\left\{x_n,\ y_n\right\} = \left\{\xi + \frac{2^{n-1} - 1}{2^{n-1}},\ \xi + \frac{2^{n-1} - 1}{2^{n-1}}\right\}. \tag{57}$$

If situation (57) is optimal, then, in the n-th row of matrix (11), inequality

$$k_{nj} = K\left(x_n,\ y_j\right) = x_n - y_j - x_n y_j \geqslant 0 \ \ \forall\, y_j < x_n \text{ by } j = \overline{1,\ n-1} \tag{58}$$

must hold. From inequality (58) it follows that

$$\frac{x_n}{1 + x_n} \geqslant y_j \ \ \forall\, y_j < x_n \text{ by } j = \overline{1,\ n-1}. \tag{59}$$

As

$$y_j \leqslant \xi + \frac{2^{n-2}-1}{2^{n-2}} = y_{n-1} < \xi + \frac{2^{n-1}-1}{2^{n-1}} = x_n, \tag{60}$$

then inequality (59) is transformed into

$$\left(\xi + \frac{2^{n-1}-1}{2^{n-1}}\right) \cdot \frac{1}{1+\xi+\dfrac{2^{n-1}-1}{2^{n-1}}} \geqslant \xi + \frac{2^{n-2}-1}{2^{n-2}},$$

$$\frac{\xi \cdot 2^{n-1} + 2^{n-1} - 1}{2^n + \xi \cdot 2^{n-1} - 1} \geqslant \frac{\xi \cdot 2^{n-2} + 2^{n-2} - 1}{2^{n-2}},$$

$$\frac{\left(\xi \cdot 2^{n-1} + 2^{n-1} - 1\right) \cdot 2^{n-2} - \left(2^n + \xi \cdot 2^{n-1} - 1\right) \cdot \left(\xi \cdot 2^{n-2} + 2^{n-2} - 1\right)}{\left(2^n + \xi \cdot 2^{n-1} - 1\right) \cdot 2^{n-2}} \geqslant 0. \tag{61}$$

It is clear that $2^n + \xi \cdot 2^{n-1} - 1 > 0$ in the denominator of the fraction in (61), so inequality (61) holds as the numerator of the fraction in (61)

$$\begin{gathered}
\xi \cdot 2^{2n-3} + 2^{2n-3} - 2^{n-2} - \\
-\left(\xi \cdot 2^{2n-2} + \xi^2 \cdot 2^{2n-3} - \xi \cdot 2^{n-2} + 2^{2n-2} + \xi \cdot 2^{2n-3} - \right. \\
\left. -2^{n-2} - 2^n - \xi \cdot 2^{n-1} + 1\right) = \\
= -\xi^2 \cdot 2^{2n-3} + \xi \cdot \left(2^{2n-3} - 2^{2n-2} + 2^{n-2} - 2^{2n-3} + 2^{n-1}\right) + \\
+ 2^{2n-3} - 2^{n-2} - 2^{2n-2} + 2^{n-2} + 2^n - 1 = \\
= -\xi^2 \cdot 2^{2n-3} + \xi \cdot 2^{n-2} \cdot (3 - 2^n) - 2^{2n-3} + 2^n - 1 \geqslant 0.
\end{gathered} \tag{62}$$

The discriminant of the respective quadratic equation

$$-\xi^2 \cdot 2^{2n-3} + \xi \cdot 2^{n-2} \cdot (3 - 2^n) - 2^{2n-3} + 2^n - 1 = 0 \tag{63}$$

is

$$\begin{gathered}
D = 2^{2n-4} \cdot (3 - 2^n)^2 + 4 \cdot 2^{2n-3} \cdot \left(-2^{2n-3} + 2^n - 1\right) = \\
= 2^{2n-4} \cdot \left(9 - 6 \cdot 2^n + 2^{2n} - 8 \cdot 2^{2n-3} + 8 \cdot 2^n - 8\right) = 2^{2n-4} \cdot \left(1 + 2^{n+1}\right),
\end{gathered}$$

whence (63) holds by

$$\begin{gathered}
\xi = \frac{-2^{n-2} \cdot (3 - 2^n) - \sqrt{2^{2n-4} \cdot (1 + 2^{n+1})}}{-2^{2n-2}} = \\
= \frac{2^{n-2} \cdot (3 - 2^n) + 2^{n-2} \cdot \sqrt{(1 + 2^{n+1})}}{2^{2n-2}} = \frac{3 - 2^n + \sqrt{1 + 2^{n+1}}}{2^n}
\end{gathered}$$

and

$$\xi = \frac{3 - 2^n - \sqrt{1 + 2^{n+1}}}{2^n}.$$

So, (62) is true by

$$\xi \in \left[\frac{3 - 2^n - \sqrt{1 + 2^{n+1}}}{2^n};\ \frac{3 - 2^n + \sqrt{1 + 2^{n+1}}}{2^n}\right]. \tag{64}$$

At $n = 4$,

$$\frac{3 - 2^4 + \sqrt{1 + 2^5}}{2^4} = \frac{\sqrt{33} - 13}{16} \in (-0.5;\ -0.45)\,,$$

but

$$k_{43} = K\,(x_4,\ y_3) = K\left(\xi + \frac{7}{8},\ \xi + \frac{3}{4}\right) =$$
$$= \xi + \frac{7}{8} - \left(\xi + \frac{3}{4}\right) - \left(\xi + \frac{7}{8}\right)\left(\xi + \frac{3}{4}\right) =$$
$$= \frac{1}{8} - \xi^2 - \xi \cdot \frac{13}{8} - \frac{21}{32} = -\xi^2 - \xi \cdot \frac{13}{8} - \frac{17}{32} \geqslant 0$$
$$\text{by } \ \xi \in \left[-\frac{\sqrt{33} + 13}{16};\ \frac{\sqrt{33} - 13}{16}\right] \tag{65}$$

and

$$k_{4N} = K\,(x_4,\ y_N) = K\left(\xi + \frac{7}{8},\ 1\right) =$$
$$= \xi + \frac{7}{8} - 1 + \xi + \frac{7}{8} = 2\xi + \frac{3}{4} \geqslant 0 \ \text{ by } \ \xi \in \left[-\frac{3}{8};\ -\frac{1}{4}\right), \tag{66}$$

where

$$\frac{\sqrt{33} - 13}{16} < -\frac{3}{8}$$

and thus

$$\left[-\frac{\sqrt{33} + 13}{16};\ \frac{\sqrt{33} - 13}{16}\right] \cap \left[-\frac{3}{8};\ -\frac{1}{4}\right) = \emptyset.$$

The latter means that inequalities (65) and (66) are impossible together, and so the fourth row of matrix (11) does not contain saddle points.

At $n = 5$,

$$\frac{3 - 2^5 - \sqrt{1 + 2^6}}{2^5} = -\frac{\sqrt{65} + 29}{32} < -1.15,$$

$$\frac{3 - 2^5 + \sqrt{1 + 2^6}}{2^5} = \frac{\sqrt{65} - 29}{32} < -0.65,$$

i. e. inequality (62) holds only at ξ for which (55) is false, or, in other words, inequality (62) is impossible for $n = 5$ and (55). Denote $b = 2^n$ for $n \geqslant 5$ and consider the right endpoint of the interval in (64) as a function of 2^n:

$$\frac{3 - 2^n + \sqrt{1 + 2^{n+1}}}{2^n} = f\,(b) = \frac{3 - b + \sqrt{1 + 2b}}{b}. \tag{67}$$

The first derivative of function (67) is

$$\frac{df(b)}{db}=\frac{-b+\dfrac{2b}{2\sqrt{1+2b}}-3+b-\sqrt{1+2b}}{b^2}=$$
$$=\frac{b-3\sqrt{1+2b}-1-2b}{b^2\sqrt{1+2b}}=-\frac{3\sqrt{1+2b}+1+b}{b^2\sqrt{1+2b}}<0,$$

so function (67) is decreasing. Therefore,

$$\max_{b\geqslant 32} f(b)=\max_{b\geqslant 32}\frac{3-b+\sqrt{1+2b}}{b}=$$
$$=\max_{n\geqslant 5}\frac{3-2^n+\sqrt{1+2^{n+1}}}{2^n}=f\left(2^5\right)=$$
$$=\frac{\sqrt{65}-29}{32}<-0.65<-\frac{1}{2}+\frac{1}{2^{N-2}}, \tag{68}$$

whence inequality (62) is impossible for $n \geqslant 5$ and (55). The latter means that situation (57) is not optimal also for $n \in \left\{\overline{5,\ N-1}\right\}$ and $N \in \mathbb{N}\backslash\{1,\ 2,\ 3,\ 4\}$.

In the last row of matrix (11),

$$k_{N,N-1}=K\left(x_N,\ y_{N-1}\right)=K\left(1,\ \xi+\frac{2^{N-2}-1}{2^{N-2}}\right)=$$
$$=1-\left(\xi+\frac{2^{N-2}-1}{2^{N-2}}\right)-\left(\xi+\frac{2^{N-2}-1}{2^{N-2}}\right)=1-2\xi-\frac{2^{N-2}-1}{2^{N-3}}=$$
$$=-1-2\xi+\frac{1}{2^{N-3}}=2\cdot\left(-\frac{1}{2}+\frac{1}{2^{N-2}}\right)-2\xi<0$$

due to (55), and thus the payoff matrix of any PDSD (10) by (8) and (11) for $N \in \mathbb{N}\backslash\{1,\ 2,\ 3,\ 4\}$ does not contain saddle points by (55). □

6. Recapitulation

In the PDSD with a positive jitter, the only optimal behavior of the duelist is to shoot at the positively ξ-jittered middle of the duel time span. This is ascertained by Theorem 1 and Theorem 2. When a negative jitter exists, it is reasonable to consider 3×3 and 4×4 PDSDs separately from bigger PDSDs. The only optimal behavior of the duelist is to shoot at the very end in the 3×3 PDSD with a negative jitter (Theorem 1). The 4×4 PDSD with a negative jitter higher than $\frac{\sqrt{17}-5}{8}$ does not have a pure strategy solution (Theorem 4). Neither do bigger PDSDs by such a negative jitter (Theorem 4). The only optimal behavior of the duelist in the 4×4 PDSD with a negative jitter higher than $-\frac{1}{4}$ and not higher than $\frac{\sqrt{17}-5}{8}$ is to shoot at the negatively $|\xi|$-jittered moment following the negatively $|\xi|$-jittered middle of the duel time span (Theorem 3). Such a behavior remains optimal for bigger PDSDs as well (Theorem 3).

Value $-\frac{1}{4}$ is the boundary case of the negative jitter, at which, as Theorem 6 asserts, the 4×4 PDSD has four optimal situations whose strategies include only the duel end moment and middle of the duel time span (the latter is not the moment following the duel beginning moment, but it is the moment following the negatively $\frac{1}{4}$-jittered middle of the duel time span). Bigger PDSDs, however, have the single optimal situation at the middle of the duel time span (Theorem 5). Here, the assertion of Theorem 3 might have been modified in order to consider the closed interval between $-\frac{1}{4}$ and $\frac{\sqrt{17}-5}{8}$, by considering only 5×5 PDSDs and bigger, and thus to merge with Theorem 5.

The only optimal behavior of the duelist is to shoot at the very end in the 4×4 PDSD with a negative jitter higher than $-\frac{1}{2}$ and lower than $-\frac{1}{4}$ (Theorem 8). Such a behavior remains optimal for bigger $N \times N$ PDSDs with a negative jitter higher than $-\frac{1}{2}$ and not higher than $-\frac{1}{2} + \frac{1}{2^{N-2}}$ (Theorem 7). Such PDSDs do not have pure strategy solutions when a negative jitter falls between $-\frac{1}{2} + \frac{1}{2^{N-2}}$ and $-\frac{1}{4}$ (Theorem 9).

So, the positive jitter does not affect the possibility of implementing the best decision in a single action (or, in terms of the duel, in a single shot). In this case, all the possible shooting moments followed by the duel beginning moment are shifted towards the duel end moment. The negative jitter does not effect the 3×3 PDSD at all, but it affects the 4×4 PDSD at a lesser negative jitter, when its magnitude is below $\frac{5-\sqrt{17}}{8}$ (Figure 1). Nevertheless, the relative interval of the pure strategy solution nonexistence in the 4×4 PDSD with a negative jitter is narrower than the half-open interval between $-\frac{1}{2}$ and $\frac{\sqrt{17}-5}{8}$, at which the 4×4 PDSD is solved in pure strategies.

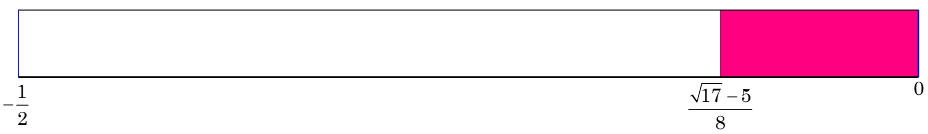

Figure 1: The relative interval of the pure strategy solution nonexistence in the 4×4 PDSD with a negative jitter

Bigger PDSDs, in which the duelist, apart from the duel beginning and end moments, possesses no fewer than three possible shooting moments, are affected to a more considerable extent. The negative jitter splits the open interval between $-\frac{1}{2}$

and 0 into four subintervals (Figure 2), at two of which an $N \times N$ PDSD has a single

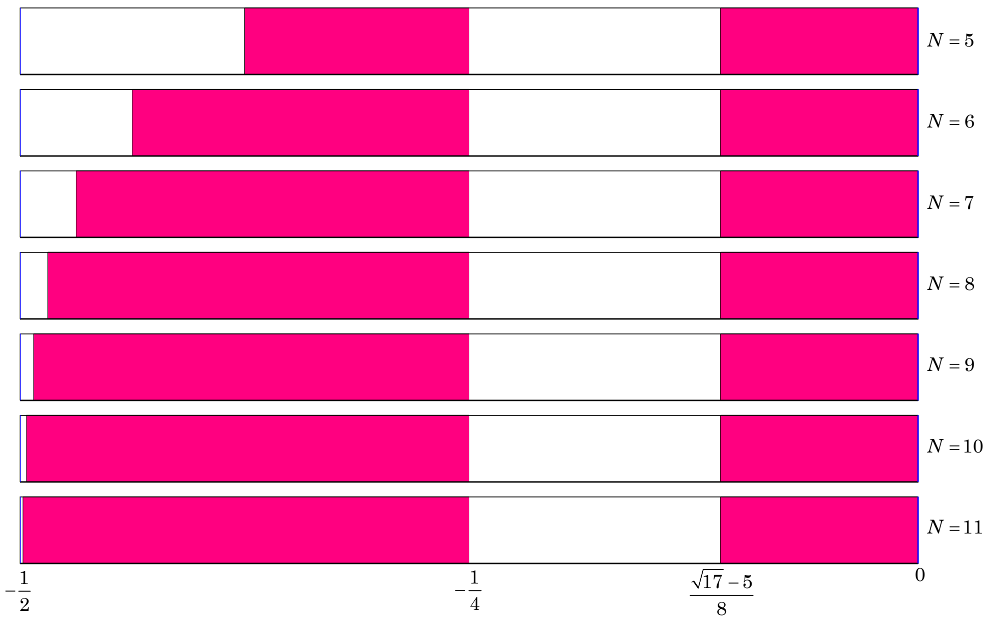

Figure 2: The change of the two relative intervals of the pure strategy solution nonexistence in $N \times N$ PDSDs with a negative jitter by $N = \overline{5,\ 11}$ (as the PDSD gets bigger)

optimal situation. Namely, if

$$\xi \in \left(-\frac{1}{2};\ -\frac{1}{2} + \frac{1}{2^{N-2}}\right] \cup \left[-\frac{1}{4};\ \frac{\sqrt{17}-5}{8}\right] \quad \text{for} \quad N \in \mathbb{N} \backslash \{1,\ 2,\ 3,\ 4\}\,, \tag{69}$$

then the $N \times N$ PDSD has a single optimal situation. As the PDSD gets bigger, the leftmost interval in (69) fades away.

7. Discussion and conclusion

The jitter is a substantially important component of a game-of-timing model that reflects imperfection of time setting and measurements. The duelists' accuracy functions presumed to be linear are hardly identical in practical applications as well, but their identity is attained on average. However, the considered shooting uniform jitter is just the first step in studying game-of-timing models with imperfection, where the symmetry is still maintained. Subsequently, the jitter may be considered non-uniform, with probably known statistical properties.

The importance of possessing an optimal pure strategy is hard to overestimate.

In real-world applications, it allows almost instantly implementing or starting to implement the best decision, unlike a mixed strategy requiring long-run repetitions of the game conditions without deviations. Pure strategy solutions in the PDSD with identical linear accuracy functions are guaranteed only for positive jitter. An exception from the rule exists for the trivial case, where any 3×3 PDSD is solved in pure strategies, whichever sign and magnitude of the jitter are.

The 4×4 PDSD does not have a pure strategy solution only if a negative jitter is higher than $\frac{\sqrt{17}-5}{8}$. Bigger PDSDs, in addition to this rule, have another open interval of the pure strategy solution nonexistence — it is (55) for $N \times N$ PDSDs, $N \in \mathbb{N} \backslash \{1, 2, 3, 4\}$. The latter interval gets wider as N increases, approaching to interval $\left(-\frac{1}{2}; -\frac{1}{4}\right)$ on the left endpoint from the right. The narrowest interval of pure strategy solution nonexistence at $\xi < -\frac{1}{4}$ has the length of $\frac{1}{8}$ (it is when $N = 5$), which is more than 1.14 times longer than the low-negative-jitter interval of pure strategy solution nonexistence (31). As N increases (i. e., the PDSD gets bigger), the ultimately-high-negative-jitter interval (43) gets narrower, making the only optimal behavior of the duelist to shoot at the very end of the duel less probable compared to other intervals of negative jitter.

References

[1] E.N. Barron, *Game Theory : An Introduction* (2nd ed.), Wiley, Hoboken, New Jersey, 2013.

[2] R.A. Epstein, *The Theory of Gambling and Statistical Logic* (2nd ed.), Academic Press, Burlington, Massachusetts, 2013.

[3] S. Karlin, *The Theory of Infinite Games. Mathematical Methods and Theory in Games, Programming, and Economics*, Pergamon, London — Paris, 1959.

[4] J.P. Lang, G. Kimeldorf, *Duels with continuous firing*, Management Science 2 (4) (1975) 470 — 476.

[5] M.J. Osborne, *An Introduction to Game Theory*, Oxford University Press, Oxford, UK, 2003.

[6] J.F. Reinganum, *Chapter 14 — The Timing of Innovation: Research, Development, and Diffusion*, in: R. Willig, R. Schmalensee (Eds.), Handbook of Industrial Organization, Elsevier, North-Holland, Volume 1, 1989, pp. 849 — 908.

[7] V.V. Romanuke, *Theory of Antagonistic Games*, New World — 2000, Lviv, 2010.

[8] V.V. Romanuke, *Fast solution of the discrete noiseless duel with the nonlinear scale on the linear accuracy functions*, Herald of Khmelnytskyi National University. Economical Sciences 5 (4) (2010) 61 — 66.

[9] V.V. Romanuke, *Discrete noiseless duel with a skewsymmetric payoff function on the unit square for models of socioeconomic competitive processes with a finite number of pure strategies*, Cybernetics and Systems Analysis 47 (5) (2011) 818 — 826.

[10] V.V. Romanuke, *Convergence and estimation of the process of computer implementation of the optimality principle in matrix games with apparent play horizon*, Journal of Automation and Information Sciences 45 (10) (2013) 49 — 56.

[11] V.V. Romanuke, *Zero-sum games on a product of staircase-function finite spaces*, Journal of Mathematics and Applications 44 (2021) 75 — 91.

[12] V.V. Romanuke, *Finite uniform approximation of two-person games defined on a product of staircase-function infinite spaces*, International Journal of Approximate Reasoning 145 (2022) 139 — 162.

[13] V.V. Romanuke, *Pure strategy saddle points in the generalized progressive discrete silent duel with identical linear accuracy functions*, Journal of Information and Organizational Sciences 48 (1) (2024).

[14] S. Siddharta, M. Shubik, *A model of a sudden death field-goal football game as a sequential duel*, Mathematical Social Sciences 15 (3) (1988) 205 — 215.

[15] J.-H. Steg, *On identifying subgame-perfect equilibrium outcomes for timing games*, Games and Economic Behavior 135 (2022) 74 — 78.

[16] Y. Teraoka, *A two-person game of timing with random arrival time of the object*, Mathematica Japonica 24 (1979) 427 — 438.

[17] S. Trybuła, *Solution of a silent duel with arbitrary motion and with arbitrary accuracy functions*, Optimization 27 (1993) 151 — 172.

[18] T. Vincent, J. Brown, *Evolutionary Game Theory, Natural Selection, and Darwinian Dynamics*, Cambridge University Press, Cambridge, UK, 2005.

[19] N.N. Vorob'yov, *Game Theory for Economists-Cyberneticists*, Nauka, Moscow, 1985.

DOI: 10.7862/rf.2024.6

Vadim Romanuke
email: `v.romanuke@amw.gdynia.pl`
ORCID: 0000-0003-3543-3087
Faculty of Mechanical and Electrical Engineering
Polish Naval Academy
Gdynia
POLAND

Received 15.03.2024 *Accepted 10.05.2024*