

Nonlinear Fractional Differential Equations with Non-Instantaneous Impulses in Banach Spaces

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ABSTRACT: This paper is devoted to study the existence of solutions for a class of initial value problems for non-instantaneous impulsive fractional differential equations involving the Caputo fractional derivative in a Banach space. The arguments are based upon Mönch's fixed point theorem and the technique of measures of noncompactness.

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1. Introduction

The theory of fractional differential equations is an important branch of differential equation theory, which has an extensive physical, chemical, biological, and engineering background, and hence has been emerging as an important area of investigation in the last few decades; see the monographs of Abbas *et al.* [3, 4], Kilbas *et al.* [18], Podlubny [23], and Zhou [25], and the references therein.

On the other hand, the theory of impulsive differential equations has undergone rapid development over the years and played a very important role in modern applied mathematical models of real processes rising in phenomena studied in physics, population dynamics, chemical technology, biotechnology and economics; see for instance the monographs by Bainov and Simeonov [12], Benchohra *et al.* [13], Lakshmikantham *et al.* [19], and Samoilenko and Perestyuk [24] and references therein. Moreover,

impulsive differential equations present a natural framework for mathematical modeling of several real-world problems. In pharmacotherapy, instantaneous impulses cannot describe the dynamics of certain evolution processes. For example, when one considers the hemodynamic equilibrium of a person, the introduction of the drugs in the bloodstream and the consequent absorption for the body are a gradual and continuous process. In [1, 2, 5, 16, 22] the authors studied some new classes of abstract impulsive differential equations with not instantaneous impulses.

However, the theory for fractional differential equations in Banach spaces has yet been sufficiently developed. Recently, Benchohra *et al.* [14] applied the measure of noncompactness to a class of Caputo fractional differential equations of order $r \in (0, 1]$ in a Banach space. Let E be a Banach space with norm $\|\cdot\|$.

In this paper, we study the following initial value problem (IVP for short), for fractional order differential equations

$${}^c D^r y(t) = f(t, y(t)), \quad \text{for a.e. } t \in (s_k, t_{k+1}], \quad k = 0, \dots, m, \quad 0 < r \leq 1, \quad (1)$$

$$y(t) = g_k(t, y(t)), \quad t \in (t_k, s_k], \quad k = 1, \dots, m, \quad (2)$$

$$y(0) = y_0, \quad (3)$$

where ${}^c D^r$ is the Caputo fractional derivative, $f : J \times E \rightarrow E$, $g_k : (t_k, s_k] \times E \rightarrow E$, $k = 1, \dots, m$, are given functions, $J = [0, T]$ and $y_0 \in E$, $0 = s_0 < t_1 < s_1 < \dots < t_m < s_m < t_{m+1} = T$.

To our knowledge no paper has been considered for non-instantaneous impulsive fractional differential equations in abstract spaces. This paper fills the gap in the literature. To investigate the existence of solutions of the problem above, we use Mönch's fixed point theorem combined with the technique of measures of noncompactness, which is an important method for seeking solutions of differential equations. See Akhmerov *et al.* [7], Álvarez [8], Banaś *et al.* [9, 10, 11], Guo *et al.* [15], Mönch [20], Mönch and Von Harten [21].

2. Preliminaries

In this section, we first state the following definitions, lemmas and some notation. By $C(J, E)$ we denote the Banach space of all continuous functions from J into E with the norm

$$\|y\|_\infty = \sup\{\|y(t)\| : t \in J\}.$$

Let $L^1(J, E)$ be the Banach space of measurable functions $y : J \rightarrow E$ which are Bochner integrable, equipped with the norm

$$\|y\|_{L^1} = \int_0^T \|y(t)\| dt.$$

$PC(J, E) = \{y : J \rightarrow E : y \in C((t_k, t_{k+1}], E), k = 0, \dots, m \text{ and there exist } y(t_k^-)$
 and $y(t_k^+), k = 1, \dots, m \text{ with } y(t_k^-) = y(t_k)\}$.

$PC(J, E)$ is a Banach space with the norm

$$\|y\|_{PC} = \sup_{t \in J} \|y(t)\|.$$

Set

$$J' = J \setminus \cup_{k=1}^m (t_k, s_k].$$

Moreover, for a given set V of functions $v : J \rightarrow E$, let us denote by

$$V(t) = \{v(t), v \in V\}, t \in J$$

and

$$V(J) = \{v(t), v \in V, t \in J\}.$$

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness.

Definition 2.1. ([9]). Let X be a Banach space and Ω_X the bounded subsets of X . The Kuratowski measure of noncompactness is the map $\alpha : \Omega_X \rightarrow [0, \infty]$ defined by

$$\alpha(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^n B_i \text{ and } \text{diam}(B_i) \leq \epsilon\}; \text{ here } B \in \Omega_X.$$

Properties: The Kuratowski measure of noncompactness satisfies the following properties (for more details see [9])

- (a) $\alpha(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact).
- (b) $\alpha(B) = \alpha(\overline{B})$.
- (c) $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$.
- (d) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$.
- (e) $\alpha(cB) = |c|\alpha(B); c \in \mathbb{R}$.
- (f) $\alpha(\text{conv}B) = \alpha(B)$.

For completeness we recall the definition of Caputo derivative of fractional order.

Definition 2.2. ([18]). The fractional (arbitrary) order integral of the function $h \in L^1([0, T], E)$ of order $r \in \mathbb{R}_+$ is defined by

$$I^r h(t) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} h(s) ds, \text{ for a.e. } t \in [0, T],$$

where Γ is the Euler gamma function defined by $\Gamma(r) = \int_0^\infty t^{r-1} e^{-t} dt, r > 0$.

Definition 2.3. ([18]). For a function $h \in AC^n(J, E)$, the Caputo fractional-order derivative of order r of h is defined by

$$({}^c D_0^r h)(t) = \frac{1}{\Gamma(n-r)} \int_0^t (t-s)^{n-r-1} h^{(n)}(s) ds, \quad \text{for a.e. } t \in [0, T],$$

where $n = [r] + 1$.

We need the following auxiliary lemmas ([18]).

Lemma 2.4. Let $r > 0$ and $h \in AC^n(J, E)$. Then the differential equation

$${}^c D_0^r h(t) = 0, \quad \text{for a.e. } t \in J$$

has solutions $h(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}$, $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$, $n = [r] + 1$.

Lemma 2.5. Let $r > 0$ and $h \in AC^n(J, E)$. Then

$$I^r {}^c D_0^r h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}, \quad \text{for a.e. } t \in J$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$, $n = [r] + 1$.

Definition 2.6. A map is said to be Carathéodory if

i $t \rightarrow f(t, u)$ is measurable for each $u \in E$.

ii $u \rightarrow F(t, u)$ is continuous for almost all $t \in J$.

For our purpose we will only need the following fixed point theorem, and the important Lemma.

Theorem 2.7. ([6, 20]) (Mönch's fixed point theorem). Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication

$$V = \overline{\text{conv}}N(V) \text{ or } V = N(V) \cup \{0\} \Rightarrow \alpha(V) = 0$$

holds for every subset V of D , then N has a fixed point.

Lemma 2.8. ([15]) If $V \subset C(J; E)$ is a bounded and equicontinuous set, then

(i) the function $t \rightarrow \alpha(V(t))$ is continuous on J , and

$$\alpha_c(V) = \sup_{0 \leq t \leq T} \alpha(V(t)).$$

(ii) $\alpha(\int_0^T x(s) ds : x \in V) \leq \int_0^T \alpha(V(s)) ds$,

where

$$V(s) = \{x(s) : x \in V\}, \quad s \in J.$$

3. Existence of Solutions

First of all, we define what we mean by a solution of the IVP (1)-(3).

Definition 3.1. A function $y \in PC(J, E) \cap AC(J', E)$ is said to be a solution of (1)-(3) if y satisfies $y(0) = y_0$, ${}^cD^r y(t) = f(t, y(t))$, for a.e. $t \in (s_k, t_{k+1}]$, and each $k = 0, \dots, m$, and $y(t) = g_k(t, y(t))$, for all $t \in (t_k, s_k]$, and every $k = 1, \dots, m$,

To prove the existence of solutions to (1)-(3), we need the following auxiliary lemmas.

Lemma 3.2. Let $0 < r \leq 1$ and let $h : J \rightarrow E$ be integrable. Then linear problem

$${}^cD^r y(t) = h(t), \text{ for each } t \in J_k := (s_k, t_{k+1}], \quad k = 0, \dots, m, \quad (4)$$

$$y(t) = g_k(t), \text{ for each } t \in J'_k := (t_k, s_k] \quad k = 1, \dots, m, \quad (5)$$

$$y(0) = y_0 \quad (6)$$

has a unique solution which is given by:

$$y(t) = \begin{cases} y_0 + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} h(s) ds, & \text{if } t \in [0, t_1], \\ g_k(t), & \text{if } t \in J'_k \quad k = 1, \dots, m, \\ g_k(s_k) + \frac{1}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} h(s) ds, & \text{if } t \in J_k \quad k = 1, \dots, m. \end{cases} \quad (7)$$

Proof. Assume that y satisfies (4)-(6).

If $t \in [0, t_1]$ then

$${}^cD^r y(t) = h(t).$$

Lemma 2.5 implies

$$y(t) = y_0 + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} h(s) ds.$$

If $t \in J'_1 = (t_1, s_1]$ we have $y(t) = g_1(t)$.

If $t \in J_1 = (s_1, t_2]$, then Lemma 2.5 implies

$$\begin{aligned} y(t) &= y(s_1^+) + \frac{1}{\Gamma(r)} \int_{s_1}^t (t-s)^{r-1} h(s) ds \\ &= g_1(s_1) + \frac{1}{\Gamma(r)} \int_{s_1}^t (t-s)^{r-1} h(s) ds. \end{aligned}$$

If $t \in J'_2 = (t_2, s_2]$ we have $y(t) = g_2(t)$.

If $t \in J_2 = (s_2, t_3]$ then again Lemma 2.5 implies

$$\begin{aligned} y(t) &= y(s_2^+) + \frac{1}{\Gamma(r)} \int_{s_2}^t (t-s)^{r-1} h(s) ds \\ &= g_2(s_2) + \frac{1}{\Gamma(r)} \int_{s_2}^t (t-s)^{r-1} h(s) ds. \end{aligned}$$

If $t \in J'_k = (t_k, s_k]$ we have $y(t) = g_k(t)$.

If $t \in J_k = (s_k, t_{k+1}]$ then Lemma 2.5 implies

$$\begin{aligned} y(t) &= y(s_k^+) + \frac{1}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} h(s) ds \\ &= g_k(s_k) + \frac{1}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} h(s) ds. \end{aligned}$$

Conversely, assume that y satisfies equation (7).

If $t \in [0, t_1]$, then $y(0) = y_0$ and, using the fact that ${}^c D^r$ is the left inverse of I^r , we get

$${}^c D^r y(t) = h(t), \text{ for each } t \in (0, t_1].$$

If $t \in J_k := (s_k, t_{k+1}]$, $k = 1, \dots, m$, and using the fact that ${}^c D^r C = 0$, where C is a constant, we get

$${}^c D^r y(t) = h(t), \text{ for each } t \in J_k := (s_k, t_{k+1}], \quad k = 1, \dots, m.$$

Also, we have easily that

$$y(t) = g_k(t), \text{ for each } t \in J'_k := (t_k, s_k], \quad k = 1, \dots, m.$$

□

We are now in a position to state and prove our existence result for the problem (1)–(3) based on Mönch's fixed point. Let us list some conditions on the functions involved in the IVP (1)–(3).

(H1) The function $f : J \times E \rightarrow E$ satisfies the Carathéodory conditions.

(H2) There exists $p \in C(J, \mathbb{R}_+)$ such that

$$\|f(t, y)\| \leq p(t)\|y\| \text{ for any } y \in E \text{ and } t \in J.$$

(H3) g_k are uniformly continuous functions and there exists $c_k \in C(J, \mathbb{R}_+)$ such that

$$\|g_k(t, y)\| \leq c_k(t)\|y\|, \text{ for each } y \in E \text{ and } t \in J, \quad k = 1, \dots, m.$$

(H4) For each bounded set $B \subset E$ we have

$$\alpha(g_k(t, B)) \leq c_k(t)\alpha(B), \quad t \in J.$$

(H5) For each bounded set $B \subset E$ we have

$$\alpha(f(t, B)) \leq p(t)\alpha(B), \quad t \in J.$$

Let

$$p^* = \sup_{t \in J} p(t), \quad c^* = \max_{k=1, \dots, m} (\sup_{t \in J} c_k(t)).$$

Theorem 3.3. *Assume that assumptions (H1)-(H5) hold. If*

$$\frac{p^* T^r}{\Gamma(r+1)} + c^* < 1, \quad (8)$$

then the IVP (1)–(3) has at least one solution J .

Proof. Transform the problem (1)–(3) into a fixed point problem. Consider the operator $N : PC(J, E) \rightarrow PC(J, E)$ defined by

$$N(y)(t) = \begin{cases} y_0 + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f(s, y(s)) ds, & \text{if } t \in [0, t_1], \\ g_k(t, y(t)), & \text{if } t \in J'_k := (t_k, s_k], \\ g_k(s_k, y(s_k)) + \frac{1}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} f(s, y(s)) ds, & \text{if } t \in J_k := (s_k, t_{k+1}]. \end{cases} \quad (9)$$

Clearly, the fixed points of operator N are solutions of problem (1)–(3).

Let

$$r_0 \geq \frac{\|y_0\|}{1 - \frac{p^* T^r}{\Gamma(r+1)} - c^*}, \quad (10)$$

and consider the set

$$D_{r_0} = \{y \in PC(J, E) : \|y\|_\infty \leq r_0\}.$$

Clearly, the subset D_{r_0} is closed, bounded and convex. We shall show that N satisfies the assumptions of Theorem 2.7. The proof will be given in a couple of steps.

Step 1: N is continuous.

Let $\{u_n\}$ be a sequence such that $u_n \rightarrow u$ in $PC(J, E)$. Then

for $t \in J_k$, we have

$$\begin{aligned} \|N(y_n)(t) - N(y)(t)\| &\leq \|g_k(t, y_n(t)) - g_k(t, y(t))\| \\ &\quad + \frac{1}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} \|f(s, y_n(s)) - f(s, y(s))\| ds, \end{aligned}$$

for $t \in [0, t_1]$, we have

$$\|N(y_n)(t) - N(y)(t)\| \leq \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} \|f(s, y_n(s)) - f(s, y(s))\| ds,$$

and for $t \in J'_k$, we have

$$\|N(u_n)(t) - N(u)(t)\| \leq \|g_k(t, y_n(t)) - g_k(t, y(t))\|.$$

Since g_k is continuous and f is of Carathéodory type, the Lebesgue dominated convergence theorem implies

$$\|N(u_n) - N(u)\|_\infty \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Consequently, N is continuous.

Step 2: N maps D_{r_0} into itself.

For each $y \in D_{r_0}$, by (H2), (H3) and (10) we have for each $t \in J$,

$$\begin{aligned} \|N(y)(t)\| &\leq \|g_k(t, y(t))\| + \|y_0\| + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} \|f(s, y(s))\| ds \\ &\leq c_k \|y(t)\| + \|y_0\| + \frac{1}{\Gamma(r)} \int_{s_k}^{t_{k+1}} (t-s)^{r-1} p(s) \|y(s)\| ds \\ &\leq \|y_0\| + r_0 \left(\frac{p^* T^r}{\Gamma(r+1)} + c^* \right) \\ &\leq r_0. \end{aligned}$$

Step 3: $N(D_{r_0})$ is bounded and equicontinuous.

By Step 2, it is obvious that $N(D_{r_0}) \subset PC(J, E)$ is bounded.

For the equicontinuous of $N(D_{r_0})$, let $\tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$ and $y \in D_{r_0}$. Then, for $\tau_1, \tau_2 \in J_k$, we have

$$\begin{aligned} \|N(y)(\tau_2) - N(y)(\tau_1)\| &= \frac{1}{\Gamma(r)} \int_{\tau_1}^{\tau_2} |(\tau_2 - s)^{r-1} - (\tau_1 - s)^{r-1}| \|f(s, y(s))\| ds \\ &\leq 2 \frac{r_0 p^*}{\Gamma(r+1)} [\tau_2^r - \tau_1^r], \end{aligned}$$

for $\tau_1, \tau_2 \in [0, t_1]$, we have

$$\begin{aligned} \|N(y)(\tau_2) - N(y)(\tau_1)\| &= \frac{1}{\Gamma(r)} \int_{\tau_1}^{\tau_2} |(\tau_2 - s)^{r-1} - (\tau_1 - s)^{r-1}| \|f(s, y(s))\| ds. \\ &\leq 2 \frac{r_0 p^*}{\Gamma(r+1)} [\tau_2^r - \tau_1^r], \end{aligned}$$

and for $\tau_1, \tau_2 \in J'_k$, we have

$$\|N(y)(\tau_2) - N(y)(\tau_1)\| = \|g_k(\tau_2, y(\tau_2)) - g_k(\tau_1, y(\tau_1))\|.$$

As $\tau_1 \rightarrow \tau_2$, the right-hand side of the above inequality tends to zero.

Now let V be a subset of D_{r_0} such that $V \subset \overline{\text{conv}}(N(V) \cup \{0\})$. Then V is bounded and equicontinuous and therefore the function $t \rightarrow v(t) = \alpha(V(t))$ is continuous on J . By (H4), (H5), Lemma 2.8 and the properties of the measure α we have for each $t \in J$

$$\begin{aligned} v(t) &\leq \alpha(N(V)(t) \cup \{0\}) \\ &\leq \alpha(N(V)(t)). \end{aligned}$$

If $t \in J_k$,

$$\begin{aligned} v(t) &\leq \alpha(g_k(s_k, V(s_k))) + \frac{1}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} f(s, V(s)) ds \\ &\leq c_k(t) \alpha(V(s)) + \frac{1}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} p(t) \alpha(V(s)) ds \\ &\leq c_k(t) v(s) + \frac{1}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} p(t) v(s) ds \\ &\leq \|v\|_\infty \left(c^* + \frac{p^* T^r}{\Gamma(r+1)} \right), \end{aligned}$$

if $t \in [0, t_1]$

$$\begin{aligned} v(t) &\leq \alpha\left(\frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f(s, V(s)) ds\right) \\ &\leq \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} p(t) \alpha(V(s)) ds \\ &\leq \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} p(t) v(s) ds \\ &\leq \|v\|_\infty \left(\frac{p^* T^r}{\Gamma(r+1)} \right) \\ &\leq \|v\|_\infty \left(c^* + \frac{p^* T^r}{\Gamma(r+1)} \right), \end{aligned}$$

if $t \in J'_k$

$$\begin{aligned} v(t) &\leq \alpha(g_k(s_k, V(s_k))) \\ &\leq c_k(t) \alpha(V(s)) \\ &\leq c_k(t) v(s) \\ &\leq \|v\|_\infty c^* \\ &\leq \|v\|_\infty \left(c^* + \frac{p^* T^r}{\Gamma(r+1)} \right). \end{aligned}$$

This means that

$$\|v\|_\infty \left[1 - \left(c^* + \frac{p^* T^r}{\Gamma(r+1)} \right) \right] \leq 0.$$

By (8) it follows that $\|v\|_\infty = 0$; that is, $v(t) = 0$ for each $t \in J$, and then $V(t)$ is relatively compact in E . In view of the Ascoli–Arzela theorem, V is relatively compact in D_{r_0} . Applying now Theorem 2.7 we conclude that N has a fixed point which is a solution of the problem (1)-(3). \square

4. An Example

Let us consider the following infinite system of impulsive fractional initial value problem,

$${}^c D^{\frac{1}{2}} y_n(t) = \frac{1}{9+n+e^t} \ln(1+|y_n(t)|), \text{ for a.e. } t \in \left(0, \frac{1}{3}\right] \cup \left(\frac{1}{2}, 1\right], \quad (11)$$

$$y_n(t) = \frac{1}{4+n+e^t} \sin |y_n(t)|, \quad t \in \left(\frac{1}{3}, \frac{1}{2}\right], \quad (12)$$

$$y_n(0) = 0. \quad (13)$$

Set

$$E = l^1 = \{y = (y_1, y_2, \dots, y_n, \dots), \sum_{n=1}^{\infty} |y_n| < \infty\},$$

E is a Banach space with the norm

$$\|y\| = \sum_{n=1}^{\infty} |y_n|.$$

Let

$$f(t, y) = (f_1(t, y), f_2(t, y), \dots, f_n(t, y), \dots),$$

$$f_n(t, y) = \frac{\ln(1+|y_n(t)|)}{9+n+e^t},$$

and

$$g_1(t, y) = (g_{11}(t, y), g_{12}(t, y), \dots, g_{1n}(t, y), \dots),$$

$$g_{1n}(t, y) = \frac{\sin |y_n(t)|}{4+n+e^t}.$$

Clearly conditions (H2) and (H3) hold with

$$p(t) = \frac{1}{9+e^t}, \text{ and } c_1(t) = \frac{1}{4+e^t}.$$

We shall check that condition (8) is satisfied with $r = \frac{1}{2}$, $T = 1$, $P^* = \frac{1}{10}$ and $c^* = \frac{1}{5}$.
Indeed

$$\left(\frac{p^* T^r}{\Gamma(r+1)} + c^* \right) = \frac{1}{5\sqrt{\pi}} + \frac{1}{5} < 1.$$

Then by Theorem 3.3 the problem (11)-(13) has at least one solution.

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