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Fixed Point Theorems for Monotone Mappings in Ordered Banach Spaces Under Weak Topology Features

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Abstract: We present several fixed point theorems for monotone nonlinear operators in ordered Banach spaces. The main assumptions of our results are formulated in terms of the weak topology. As an application, we study the existence of solutions to a class of first-order vector-valued ordinary differential equations. Our conclusions generalize many well-known results.

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1. Introduction

Fixed point theory furnishes an effective and important tool for proving theoretical as well as constructive existence for a variety of nonlinear problems arising from the mathematical modelling of real world phenomena. The usual topological fixed point methods (Schauder, Darbo, Sadovskii,...) are generally only suited to nonlinear problems with continuity and compactness. However, many problems in theory and applications have no compactness. Some attempts have been made to overcome this difficulty by using the weak topology, see [2, 3, 6, 7, 8, 9, 10, 11, 14, 34]. The interest of the weak topology is mainly due to the vital role played by weak compactness in the theory of infinite dimensional linear spaces. In particular, a Banach space X is reflexive if and only if the closed unit ball is weakly compact. Equally, fixed point theorems using the weak topology (Schauder-Tychonov, Arino-Gautier-Penot,...) are

generally only suited to nonlinear problems with weak (sequential) continuity and weak compactness. In several situations, the weak (sequential) continuity could rise several difficulties. For example, in L^1-spaces, which are the most natural functional settings of many real world problems in physics and population dynamics (notably when the unknown is a density), only linear superposition (Nemytskii) operators are weakly (sequentially) continuous [4]. To our knowledge, the first paper where the weak topology was successfully applied to fixed point theorems without requiring the weak continuity of the involved operators, was [29]. In the quoted paper, the authors used the concepts of ws-compactness and ww-compactness instead of the (sequential) weak continuity. Such concepts proved to be more effective in many practical situations especially when we work in nonreflexive Banach spaces. This fact was illustrated by proving the existence of an integrable solution for a stationary nonlinear problem arising in transport theory and kinetic of gas and in many other situations [12, 13, 16, 20, 21, 22, 29, 30].

In the present paper, we provide a new general treatment of fixed point theory of monotone mappings in ordered vector spaces. Specifically, we will show how weak topology is successfully used in conjunction with the order in fixed point problems. As the functional setting of many nonlinear problems arising from the mathematical modeling of real world phenomena is usually an ordered vector space, our approach gives an extremely powerful and direct tool to investigate the solvability of a large class of evolution equations with lack of compactness. To illustrate our results, we investigate the solvability of a class of first-order vector-valued ordinary differential equations. Before proceeding to the detailed discussion, we recall some related definitions and auxiliary results. Let X be a Banach space and let P be a subset of X. The set P is called an *order cone* if and only if:

(i) P is closed, nonempty and $P \neq \{0\}$,

(ii) $a, b \in \mathbb{R}, a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$,

(iii) $x \in P$ and $-x \in P \Rightarrow x = 0$.

An order cone permits to define a partial order in X by

$$x \leq y \text{ iff } y - x \in P.$$

Conversely, let X be a real Banach space with a partial order compatible with the algebraic operations in X, that is,

$$x \geq 0 \text{ and } \lambda \geq 0 \text{ implies } \lambda x \geq 0$$

$$x_1 \leq y_1 \text{ and } x_2 \leq y_2 \text{ implies } x_1 + x_2 \leq y_1 + y_2.$$

The *positive cone* of X is defined by

$$X^+ = \{x \in X : \ 0 \leq x\}.$$

Definition 1.1.

(i) A subset $M \subset X$ is said *order bounded* if there exist $u, v \in X$ such that $u \leq x \leq v$, for all $x \in M$.

(ii) The order cone P is called *normal* if and only if there is a number $c > 0$ such that for all $x, y \in X$ we have

$$0 \leq x \leq y \Rightarrow \|x\| \leq c\|y\|. \tag{1.1}$$

The least positive number c (if it exists) satisfying (1.1) is called a *normal constant*.

Remark 1.2. If the cone P is normal, then every order interval is norm bounded (see e.g. [23, Theorem 2.1.1]).

Remark 1.3. Let K be a compact Hausdorff space and E be an ordered Banach space with normal positive cone. We denote by $C(K, E)$ the Banach space of all continuous E -valued functions on K endowed with the usual maximum norm. Plainly $C(K, E)$ is an ordered Banach space with the natural ordering whose positive cone is given by

$$C^+(K, E) = \{f \in C(K, E) : f(x) \in E^+, \forall x \in K\}.$$

Since E^+ is normal so is $C^+(K, E)$.

The following definitions are frequently used in the sequel.

Definition 1.4. Let $M \subset X$. The operator $T\colon M \to X$ is said to be an increasing operator if $x, y \in M$, $x \leq y$ implies $Tx \leq Ty$. The operator $T\colon M \to X$ is said to be a decreasing operator if $x, y \in M$, $x \leq y$ implies $Ty \leq Tx$.

Definition 1.5. Let M be a nonempty closed subset of X. The operator $T\colon M \to X$ is said to be monotone-subcontinuous if for any monotone sequence (increasing or decreasing) (x_n) in M that converges strongly to x the sequence (Tx_n) converges weakly to Tx.

The following elementary result serves as the key tool in the proof of more sophisticated results.

Lemma 1.6. *[26] Let X be an ordered real Banach space with a normal order cone. Suppose that $\{x_n\}$ is a monotone sequence which has a subsequence $\{x_{n_k}\}$ converging weakly to x_∞. Then $\{x_n\}$ converges strongly to x_∞. Moreover, if $\{x_n\}$ is an increasing sequence, then $x_n \leq x_\infty (n = 1, 2, 3, \ldots)$; if $\{x_n\}$ is a decreasing sequence, then $x_\infty \leq x_n$ $(n = 1, 2, 3, \ldots)$.*

By a poset $F = (F, \leq)$ we mean a nonempty set F equipped with a partial ordering relation $\leq$.

Lemma 1.7. *[25, Lemma 1.1.5] Let $\{x_n\}$ be a sequence in a poset F.*

(a) *If $\{x_n\}$ is totally ordered, then it has a monotone subsequence.*

(b) *If $\{x_n\}$ is nondecreasing (resp. nonincreasing), then it has the supremum (resp. the infimum) x if and only if x is the supremum (resp. the infimum) of some of its subsequences.*

Combining Lemma 1.6 and Lemma 1.7 we obtain the following interesting result.

Lemma 1.8. *Let X be an ordered real Banach space with a normal order cone. Suppose that $\{x_n\}$ is a totally ordered sequence which is contained in a relatively weakly compact set. Then $\{x_n\}$ converges strongly in X.*

In what follows, ψ will always denote a measure of weak noncompactness (MWNC) on the Banach space X. We refer the reader to [5] for the axiomatic definition of a measure of weak noncompactness. One of the most frequently exploited measure of weak noncompactness was defined by De Blasi [15] as follows:

$$w(M) = \inf\{r > 0 : \text{ there exists } W \text{ weakly compact such that } M \subseteq W + B_r\},$$

for each bounded subset M of X; Here, B_r stands for the closed ball of X centered at origin with radius r.
The following results are crucial for our purposes. We first state a theorem of Ambrosetti type (see [31] for a proof).

Theorem 1.9. *Let E be a Banach space and let $H \subseteq C([0,T],E)$ be bounded and equicontinuous. Then the map $t \to w(H(t))$ is continuous on $[0,T]$ and*

$$w(H) = \sup_{t\in[0,T]} w(H(t)) = w(H[0,T]),$$

where $H(t) = \{h(t) : h \in H\}$ and $H[0,T] = \bigcup_{t\in[0,T]}\{h(t) : h \in H\}$.

The following Lemma is well-known (see for example [32]).

Lemma 1.10. *If $H \subseteq C([0,T],E)$ is equicontinuous and $x_0 \in C([0,T],E)$, then $\overline{co}(H \cup \{x_0\})$ is also equicontinuous in $C([0,T],E)$.*

2. Fixed point results

In this section, we prove some fixed point theorems for monotone mappings in ordered Banach spaces. Our results combine the advantages of the strong topology (i.e. the involved mappings will be continuous (or subcontinuous) with respect to the strong topology) with the advantages of the weak topology (i.e. the maps will satisfy some compactness conditions relative to the weak topology) to draw new conclusions about fixed points for a given monotone map.

Theorem 2.1. *Let X be an ordered Banach space with a normal cone P. Let $u_0, v_0 \in X$ with $u_0 < v_0$ and $A : [u_0, v_0] \to X$ be a monotone-subcontinuous increasing operator satisfying the following:*

$$u_0 \leq Au_0,\ Av_0 \leq v_0. \tag{2.1}$$

If, in addition, A verifies

$(\mathcal{P}(n_0))$: There exists an integer $n_0 \geq 1$ such that: for any monotone sequence $V = \{x_n\}$ of $[u_0, v_0]$ and any finite subset F of $[u_0, v_0]$ of cardinal n_0, we have:

$$V = F \cup A^{n_0}(V) \text{ implies } V \text{ is relatively weakly compact.}$$

Then, A has a minimal fixed point u_* and a maximal fixed point u^* in $[u_0, v_0]$ and

$$u_* = \lim_{n\to\infty} u_n \text{ and } u^* = \lim_{n\to\infty} v_n, \tag{2.2}$$

where $u_n = Au_{n-1}$ and $v_n = Av_{n-1},\ n = 1, 2, \ldots$

$$u_0 \leq u_1 \leq \cdots u_* \leq u^* \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0. \tag{2.3}$$

Proof. Let $u_n = Au_{n-1}$ and $v_n = Av_{n-1}$ for $n \geq 1$. Since A is increasing, then

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots v_n \leq \cdots v_1 \leq v_0. \tag{2.4}$$

Let $S = \{u_0, u_1, \ldots, u_n, \ldots\}$. Clearly, for any integer $k \geq 1$ we have

$$A^k(S) \cup \{u_0, u_1, \ldots, u_{k-1}\} = S.$$

From our hypotheses we know that S is relatively weakly compact. Referring to Lemma 1.8, we see that $\{u_n\}$ is convergent. Let u_* be its limit. The monotone-subcontinuity of A yields $Au_* = u_*$. Similarly, we can prove that $\{v_n\}$ converges to some u^* and $Au^* = u^*$. Finally, we prove that u^* and u_* are the maximal and minimal fixed points of A in $[u_0, v_0]$. Let $x \in [u_0, v_0]$ and $Ax = x$. Since A is increasing, it follows from $u_0 \leq x \leq v_0$ that $Au_0 \leq Ax \leq Av_0$, i.e. $u_1 \leq x \leq v_1$. Using the same argument, we get $u_2 \leq x \leq v_2$ and, in general, $u_n \leq x \leq v_n$ $(n = 1, 2, 3, \ldots)$. Now, letting n go to infinity we get $u_* \leq x \leq u^*$. □

As a convenient specialization of Theorem 2.1, we state the following.

Corollary 2.2. *Let X be an ordered Banach space with a normal cone P. Let $u_0, v_0 \in X$ with $u_0 < v_0$ and $A\colon [u_0, v_0] \to X$ be a monotone-subcontinuous increasing operator satisfying the following:*

$$u_0 \leq Au_0,\ Av_0 \leq v_0. \tag{2.5}$$

If, in addition, A verifies

$(\mathcal{P}(1))$: if $V = \{x_n\}$ is a monotone sequence of $[u_0, v_0]$ and $a \in [u_0, v_0]$, then $V = \{a\} \cup A(V)$ implies V is relatively weakly compact.
Then A has a minimal fixed point u_* and a maximal fixed point u^* in $[u_0, v_0]$ satisfying (2.2) and (2.3).

Proof. Apply Theorem 2.1 with $n_0 = 1$. □

Another consequence of Theorem 2.1 is the following. Recall that a measure of weak noncompactness ψ on a Banach space X is said to be *nonsingular* if $\psi(M \cup \{a\}) = \psi(M)$ for every $a \in X$ and every nonempty bounded subset M of X.

Corollary 2.3. *Let X be an ordered Banach space with a normal cone P and ψ be a nonsingular measure of weak noncompactness on X. Let $u_0, v_0 \in X$ with $u_0 < v_0$ and $A\colon [u_0, v_0] \to X$ be a monotone-subcontinuous increasing operator satisfying the following:*

$$u_0 \leq Au_0,\ Av_0 \leq v_0. \tag{2.6}$$

In addition, if for any $\Omega = \{u_n\} \subset [u_0, v_0]$ countable and monotone with $\psi(\Omega) \neq 0$ we have

$$\psi(A^{n_0}(\Omega)) < \psi(\Omega),$$

for some integer $n_0 \geq 1$. Then, A has a minimal fixed point u_* and a maximal fixed point u^* in $[u_0, v_0]$ satisfying (2.2) and (2.3).

Proof. By virtue of Theorem 2.1, it suffices to show that $(\mathcal{P}(n_0))$ holds true. To do this, let $V = \{x_n\}$ be a monotone sequence of $[u_0, v_0]$ and F be a finite subset of $[u_0, v_0]$ of cardinal n_0 such that $V = F \cup A^{n_0}(V)$. Since P is normal then, according to Remark 1.2, the order interval $[u_0, v_0]$ is bounded. This implies that V and $A^{n_0}(V)$ are bounded and we have $\psi(V) = \psi(F \cup A^{n_0}(V)) = \psi(A^{n_0}(V))$. Consequently, it follows from our hypotheses that $\psi(V) = 0$, which means that V is relatively weakly compact. This achieves the proof.

□

Remark 2.4. Corollary 2.3 extends [23, Theorem 3.1.1].

Corollary 2.5. *Let $u_0, v_0 \in X$ with $u_0 < v_0$ and $A\colon [u_0, v_0] \to X$ be a monotone-subcontinuous increasing operator satisfying (2.6). If P is normal and $A^{n_0}([u_0, v_0])$ is relatively weakly compact for some integer $n_0 \geq 1$, then A has a minimal fixed point u_* and a maximal fixed point u^* in $[u_0, v_0]$ satisfying (2.2) and (2.3).*

For later use, we consider the following condition.

$$(\mathcal{C})\ \begin{cases} A\colon P \to P \text{ satisfies } A^2\theta \geq \epsilon A\theta \text{ where } 0 < \epsilon < 1, \text{and for any} \\ \epsilon A\theta \leq x \leq A\theta \text{ and } \epsilon \leq t < 1, \text{ there exists } \eta = \eta(x,t) > 0, \text{ such that} \\ A(tx) \leq (t(1+\eta))^{-1} Ax. \end{cases}$$

We will need the following lemmas from [23].

Lemma 2.6. *[23, Lemma 3.2.1] Let $A\colon P \to P$ be a decreasing operator satisfying the condition $(\mathcal{C})$. If $u, v \in P$ with $Au = v$ and $Av = u$, then $u = v$.*

Lemma 2.7. *[23, Lemma 3.2.2] Let $A\colon P \to P$ be a decreasing operator satisfying the condition $(\mathcal{C})$. If $u, v \in P$ with $Au = u$ and $Av = v$, then $u = v$.*

Theorem 2.8. *Let X be an ordered Banach space with a normal cone P. Let $A\colon P \to P$ be a monotone-subcontinuous decreasing operator satisfying the conditions $(\mathcal{C})$ and $(\mathcal{P}(n_0))$ for some integer $n_0 \geq 1$. Then A has a unique fixed point u^* in P and*

$$u^* = \lim_{n\to\infty} u_n, \tag{2.7}$$

where $u_n = Au_{n-1},\ n = 1, 2, \ldots$

Proof. Keeping in mind that $A\colon P \to P$ is decreasing we easily deduce that

$$\theta = u_0 \leq u_2 \leq \cdots \leq u_{2n} \leq \cdots \leq u_{2n+1} \leq \cdots \leq u_1 = A\theta. \tag{2.8}$$

Let $S = \{u_0, u_1, \ldots, u_n, \ldots\}$. From (2.8) and the normality of P we infer that S is bounded. Clearly, for any integer $k \geq 1$ we have

$$A^k(S) \cup \{u_0, u_1, \ldots, u_{k-1}\} = S.$$

From our hypotheses we know that S is relatively weakly compact. This implies that the increasing sequence $\{u_{2n}\}$ has a weakly convergent subsequence. Referring to Lemma 1.6, we see that $\{u_{2n}\}$ is convergent. Let u_* be its limit. Similarly we can prove that the sequence $\{u_{2n+1}\}$ converges to some u^*. Taking the limit at the both sides of $u_{2n+1} = Au_{2n}$ and $u_{2n+2} = Au_{2n+1}$ and using the monotone-subcontinuity of A we get $u_* \leq u^*$, $u^* = Au_*$ and $u^* = Au_*$. Invoking Lemma 2.6 we infer that $u^* = u_*$ is a fixed point of A. The uniqueness follows from Lemma 2.7. □

As a convenient specialization of Theorem 2.8 we obtain the following result.

Corollary 2.9. *Let X be an ordered Banach space with a normal cone P and ψ be a nonsingular measure of weak noncompactness on X. Let $A\colon P \to P$ be a monotone-subcontinuous decreasing operator satisfying the condition $(\mathcal{C})$. In addition, if for any $\Omega = \{u_n\} \subset P$ countable and monotone with $\psi(\Omega) \neq 0$ we have*

$$\psi(A^{n_0}(\Omega)) < \psi(\Omega),$$

for some integer $n_0 \geq 1$, then A has a unique fixed point u^* in P and

$$u^* = \lim_{n\to\infty} u_n, \tag{2.9}$$

where $u_n = Au_{n-1},\ n = 1, 2, \ldots$

Proof. In view of Theorem 2.8, it suffices to show that A verifies $(\mathcal{P}(n_0))$. The reasoning in Corollary 2.3 yields the result. □

Remark 2.10. Theorem 2.8 and Corollary 2.9 extend [23, Theorem 3.2.1].

3. Application to differential equations

We shall use the results in previous sections to get an existence theorem for a nonlinear ODE in a Banach space. The nonlinear term satisfies an appropriate condition expressed in terms of the De Blasi measure of weak noncompactness. Let E be an ordered Banach space with a normal cone P. We consider the following initial value problem

$$u' = f(t,u) \text{ on } \ I, \ u(0) = u_0, \tag{3.1}$$

where $I = [0,1]$, $u \in C^1(I,E)$, $f \in C(I \times E, E)$. A vector-valued function $u\colon I \to E$ is said to be a solution of (3.1) on I if $u(t)$ is continuously differentiable and satisfies (3.1) on I.
In [18], Du and Lakshmikantham proved that if the problem (3.1) has a lower solution v_0 and an upper solution w_0 with $v_0 \leq w_0$, and the nonlinear term satisfies the monotonicity condition

$$f(t,y) - f(t,x) \geq -M(y-x) \text{ whenever } v_0(t) \leq x \leq y \leq w_0(t) \tag{3.2}$$

for some $M > 0$, and the compactness measure condition

$$\alpha(f(t,V)) \leq \tau\alpha(V) \tag{3.3}$$

for any $t \in I$ and any bounded subset V of E, where τ is a positive constant and $\alpha(.)$ denotes the Kuratowski measure of noncompactness in E, then the problem (3.1) has a minimal and a maximal solution between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0 respectively. When E is weakly sequentially complete, Y. Du [17] improved the result of [18] and removed the condition (3.3).
Our aim in this section is to improve and extend the aforementioned results. We will replace the noncompactness measure condition (3.3) by a weaker condition expressed in terms of the De Blasi measure of weak noncompactness. From now on, we assume the following:

(i) There exist $v_0, w_0 \in C^1(I,E)$ with $v_0(t) \leq w_0(t)$ on I such that:

$$v_0'(t) \leq f(t, v_0(t)),\ v_0(0) \leq u_0$$

$$w_0'(t) \geq f(t, w_0(t)),\ w_0(0) \geq u_0.$$

(ii) For some $M > 0$,

$$f(t,y) - f(t,x) \geq -M(y-x) \text{ whenever } v_0(t) \leq x \leq y \leq w_0(t).$$

(iii) There is a constant $\tau \geq 0$ such that for any equicontinuous monotone sequence $V = \{u_n\}$ of $[v_0, w_0]$ and for any $a, b \in [0,1]$ with $a < b$ we have

$$w(f([a,b] \times V)) \leq \tau w(V[a,b]),$$

where $f([a,b] \times V) := \{f(s, x(s)),\ a \leq s \leq b,\ x \in V\}$.

Remark 3.1. Let $g(s,x) = f(s,x) + Mx$. Then, for any monotone sequence $V = \{u_n\}$ of $[v_0, w_0]$ and for any $a, b \in [0,1]$ with $a < b$ we have

$$w(g([a,b] \times V)) \leq \mu w(V[a,b]), \tag{3.4}$$

where $\mu = \tau + M$.
Now, let $t \in [0,1]$ be fixed and let $h(s,x) = e^{-M(t-s)} g(s,x)$, for $s \in [0,t]$ and $x \in E$. It is readily verified that

$$h([0,t] \times V) \subset co\,(g([0,t] \times V) \cup \{0\})\,. \tag{3.5}$$

Combining (3.4) and (3.5) we arrive at

$$w(h([0,t] \times V)) \leq \mu w(V[0,t]), \tag{3.6}$$

where $h([0,t] \times V) := \{h(s, x(s)),\ 0 \leq s \leq t,\ x \in V\}$.

Now, we are in a position to state our main result.

Theorem 3.2. *Let assumptions* (i)–(iii) *be satisfied. Then the problem (3.1) has a maximal and a minimal solution between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0 respectively.*

Proof. We consider the equivalent modified problem

$$u' + Mu = f(t,u) + Mu \text{ on } \ I,\ u(0) = u_0, \tag{3.7}$$

which is equivalent to the problem

$$\left(e^{Mt} u\right)' = e^{Mt}\left(f(t,u) + Mu\right) \text{ on } \ I,\ u(0) = u_0. \tag{3.8}$$

Let us write (3.8) as an integral equation

$$u(t) = e^{-Mt} u_0 + \int_0^t e^{-M(t-s)}\left(f(s, u(s)) + Mu(s)\right) ds. \tag{3.9}$$

□

Define the operator A on $C(I,E)$ by

$$(Au)(t) = e^{-Mt} u_0 + \int_0^t e^{-M(t-s)}\left(f(s, u(s)) + Mu(s)\right) ds,\ t \in I. \tag{3.10}$$

It is easy to check that a fixed point of A is a solution of (3.1). We will demonstrate that A satisfies all the hypotheses of Theorem 2.1. It is apparent that A is continuous. From Hypothesis (ii) we know that A is increasing on $[v_0, w_0]$. To illustrate that $v_0 \leq Av_0$, let $k(t) = v_0'(t) + Mv_0(t)$. Clearly, $k \in C(I, E)$ and $k(t) \leq f(t, v_0(t)) + Mv_0(t),\ t \in I$. Keeping in mind the fact that $\left(e^{Mt}v_0(t)\right)' = e^{Mt}k(t)$, we deduce that for all $t \in I$ we have:

$$\begin{aligned} e^{Mt}v_0(t) &= v_0(0) + \int_0^t e^{Ms}k(s)ds \\ &\leq u_0 + \int_0^t e^{Ms}(f(s, v_0(s)) + Mv_0(s))ds. \end{aligned}$$

Accordingly, $v_0 \leq Av_0$. Similarly, we can prove that $Aw_0 \leq w_0$. We claim that for any integer $k \geq 1$ and any $V \subset [u_0, v_0]$ the set $A^k(V)$ is equicontinuous. Indeed, let $t, t_0 \in I$ with $t < t_0$ and $u \in [v_0, w_0]$. Then,

$$\begin{aligned} \|Au(t) - Au(t_0)\| \leq\ & (e^{-Mt} - e^{-Mt_0})\|u_0\| + \int_0^t (e^{-M(t-s)} - e^{-M(t_0-s)})\|g(s, u(s))\|ds \\ & + \int_t^{t_0} \|g(s, u(s))\|ds. \end{aligned}$$

For any $u \in [v_0, w_0]$, by Assumption (ii), we have

$$g(s, v_0(s)) \leq g(s, u(s)) \leq g(s, w_0(s)).$$

By the normality of the cone P, there exists $C_g > 0$ such that

$$\|g(t, u(t))\| \leq C_g,\ u \in [v_0, w_0].$$

Accordingly,

$$\begin{aligned} \|Au(t) - Au(t_0)\| \leq\ & (e^{-Mt} - e^{-Mt_0})\|u_0\| + C_g \int_0^t (e^{-M(t-s)} - e^{-M(t_0-s)})ds \\ & + C_g(t_0 - t). \end{aligned}$$

Consequently,

$$\|Au(t) - Au(t_0)\| \to 0 \ \text{ as } t \to t_0^-,$$

uniformly with respect to u. Similarly, we get

$$\|Au(t) - Au(t_0)\| \to 0 \ \text{ as } t \to t_0^+,$$

uniformly with respect to u. This proves that $A(V)$ is equicontinuous. Therefore, for any integer $k \geq 1$ the set $A^k(V)$ is equicontinuous.

Now, let $V \subset [v_0, w_0]$ and F be a finite subset of $[v_0, w_0]$ such that $V = A^k(V) \cup F$, for some integer $k \geq 1$. Since $A^k(V)$ is equicontinuous, then by invoking Lemma 1.10

we conclude that V is equicontinuous. Let h be as described in Remark 3.1, then for each $t \in I$, we have

$$\begin{aligned} w(A(V)(t)) &= w\left(\left\{e^{-Mt}u_0 + \int_0^t h(s,u(s))ds\colon u \in V\right\}\right) \\ &\leq w(t\overline{co}\{h(s,u(s))\colon u \in V,\, s \in [0,t]\}) \\ &= tw(\overline{co}\{h(s,u(s))\colon u \in V,\, s \in [0,t]\}) \\ &\leq tw(h([0,t] \times V) \\ &\leq t\mu w(V[0,t]). \end{aligned}$$

Theorem 1.9 implies (since V is equicontinuous) that

$$w(A(V)(t)) \leq t\mu w(V). \tag{3.11}$$

Using (3.11) we get

$$\begin{aligned} w(A^2(V)(t)) &= w\left(\left\{e^{-Mt}u_0 + \int_0^t h(s,u(s))ds\colon u \in A(V)\right\}\right) \\ &= w\left(\left\{\int_0^t h(s,u(s))ds\colon u \in A(V)\right\}\right). \end{aligned} \tag{3.12}$$

Fix $t \in [0,1]$. We divide the interval $[0,t]$ into m parts $0 = t_0 < t_1 < \cdots < t_m = t$ in such a way that $\Delta t_i = t_i - t_{i-1} = \frac{t}{m}$, $i = 1, \ldots, m$. For each $u \in A(V)$ we have

$$\begin{aligned} \int_0^t h(s,u(s))ds &= \sum_{i=1}^m \int_{t_{i-1}}^{t_i} h(s,u(s))ds \\ &\in \sum_{i=1}^m \Delta t_i \overline{co}\{h(s,u(s))\colon u \in A(V),\, s \in [t_{i-1},t_i]\} \\ &\subseteq \sum_{i=1}^m \Delta t_i \overline{co}(h([t_{i-1},t_i] \times A(V)). \end{aligned}$$

Using again Theorem 1.9 we infer that for each $i = 2, \ldots, m$ there is a $s_i \in [t_{i-1}, t_i]$ such that

$$\sup_{s \in [t_{i-1},t_i]} w(A(V)(s)) = w(A(V)[t_{i-1},t_i]) = w(A(V)(s_i)). \tag{3.13}$$

Consequently

$$\begin{aligned} w(\{\int_0^t h(s,x(s))ds\colon u \in A(V)\} &\leq \sum_{i=1}^m \Delta t_i w(\overline{co}(h([t_{i-1},t_i] \times A(V))) \\ &\leq \mu \sum_{i=1}^m \Delta t_i w((A(V)([t_{i-1},t_i])) \\ &\leq \mu \sum_{i=1}^m \Delta t_i w(A(V)((s_i)). \end{aligned}$$

On the other hand, if $m \to \infty$ then

$$\sum_{i=1}^{m} \Delta t_i w(A(V)((s_i)) \to \int_0^t w(A(V)(s))ds. \tag{3.14}$$

As a result,

$$w(A^2(V)(t)) \leq \frac{(\mu t)^2}{2} w(V). \tag{3.15}$$

By induction we get

$$w(A^n(V)(t)) \leq \frac{(\mu t)^n}{n!} w(V). \tag{3.16}$$

Invoking Theorem 1.9 we obtain

$$w(A^n(V)) \leq \frac{\mu^n}{n!} w(V). \tag{3.17}$$

Since $\lim_{n\to\infty} \frac{\mu^n}{n!} = 0$, we may choose n_0 as large as we please such that $\frac{\mu^{n_0}}{n_0!} < 1$.

Now, let $V \subset [v_0, w_0]$ and F be a finite subset of $[v_0, w_0]$ such that $V = A^{n_0}(V) \cup F$. Then, $w(V) = w(A^{n_0}(V) \cup F) = w(A^{n_0}(V)) \leq \frac{\mu^{n_0}}{n_0!} w(V)$. Thus, $w(V) = 0$ and therefore V is relatively weakly compact. By applying Theorem 2.1 we infer that A has a maximal and a minimal fixed points between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0 respectively. This completes the proof.

Remark 3.3. If E is weakly sequentially complete (reflexive, in particular), then the condition (iii) in Theorem 3.2 holds automatically. In fact, according to [17, Theorem 2.2] any monotone order-bounded sequence is relatively compact. Thus, Theorem 3.2 greatly improves [17, Theorem 4.1] and [18, Theorem 3.1].

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