Journal of
Mathematics
and Applications
JMA No 43, pp 123-138 (2020)

Finite Approximation of Continuous Noncooperative Two-person Games on a Product of Linear Strategy Functional Spaces

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Abstract: A method of the finite approximation of continuous noncooperative two-person games is presented. The method is based on sampling the functional spaces, which serve as the sets of pure strategies of the players. The pure strategy is a linear function of time, in which the trend-defining coefficient is variable. The spaces of the players' pure strategies are sampled uniformly so that the resulting finite game is a bimatrix game whose payoff matrices are square. The approximation procedure starts with not a great number of intervals. Then this number is gradually increased, and new, bigger, bimatrix games are solved until an acceptable solution of the bimatrix game becomes sufficiently close to the same-type solutions at the preceding iterations. The closeness is expressed as the absolute difference between the trend-defining coefficients of the strategies from the neighboring solutions. These distances should be decreasing once they are smoothed with respective polynomials of degree 2.

AMS Subject Classification: 91A05, 91A10, 65D99, 41A99.
Keywords and Phrases: Game theory; Payoff functional; Linear strategy; Continuous game; Finite approximation; Einstellung effect.

1. Introduction

Continuous noncooperative two-person games model interactions of a pair of subjects (players or persons) possessing continuums of their pure strategies [5, 10]. A specificity of such games consists in that finding and practicing a solution in mixed strategies is often intractable [11, 6, 9]. Even if a solution exists in pure strategies, it often is revealed not to be a single one. Thus, the problem of the single solution selection (or uniqueness) arises. However, even if the solution is unique, it is not guaranteed to be simultaneously profitable and symmetric [11, 9, 2, 1].

The solution search in continuous games is not a trivial task also. Analytic search generalization is possible only in special classes [10, 3]. Therefore, finite approximation of continuous noncooperative two-person games is not just preferable, but also is necessary.

2. Motivation

A special class of noncooperative two-person games is when the player's pure strategy is a time-varying function. Commonly, apart from the time, this function is determined by a few parameters (coefficients). These coefficients may vary through intervals. Therefore, the set of the player's pure strategies is a functional space. Such a game model is typical for economic interaction processes, where the player uses short-term time-varying strategies [11, 13, 12].

In the simplest case, the strategy is a linear function of time. The time interval is usually short, through which a short-term trend of economic activity is realized [11, 9]. Thus, a whole process is modeled as a series of those noncooperative games. Each game corresponds to its short term. Then, obviously, the games are required to be solved as fast as possible.

The problems of fast solution and solution uniqueness are addressed in studying finite approximations of continuous games. When the game is defined on finite-dimensional Euclidean subspaces, it can be approximated by appropriately sampling the sets of players' pure strategies [6, 7]. Then an approximating game is solved easily and faster. Besides, an approximated solution (with respect to the initial game) can be selected in order to meet demands and rules of the economic system [11, 9]. In the case when the game is defined on a product of functional spaces, a strict substantiation is required to sample the functional sets of players' pure strategies. As in the case of finite-dimensional Euclidean subspaces, this will allow sampling without significant losses.

3. Goals and tasks to be fulfilled

Due to above reasons, the goal is to develop a method of finite approximation of continuous noncooperative two-person games whose kernels are defined on a product of linear strategy functional spaces. For achieving the goal, the following tasks are to be fulfilled:

1. To formalize a continuous noncooperative two-person game whose kernel is defined on a product of linear strategy functional spaces. In such a game, the set of the player's pure strategies is a continuum of linear functions of time.

2. To formalize a method of finite approximation.
3. To discuss applicability and significance of the method.

4. A continuous noncooperative two-person game

Each of the players uses short-term time-varying strategies determined by two coefficients. The short-term trend is defined by a real-valued coefficient which is multiplied by time t. The other coefficient is presumed to be known (i. e., it is a constant). Herein, this real-valued constant is called an offset.

The pure strategy is valid on interval $[t_1;\ t_2]$ by $t_2 > t_1$, so pure strategies of the player belong to a functional space of linear functions of time:

$$L\left[t_1;\ t_2\right] \subset \mathbb{L}_2\left[t_1;\ t_2\right].$$

Denote the trend-defining coefficient of the first player by b_x, where

$$b_x \in \left[b_x^{(\min)};\ b_x^{(\max)}\right] \text{ by } b_x^{(\max)} > b_x^{(\min)}. \tag{1}$$

If the first player's offset is a_x, then its set of pure strategies is

$$X = \left\{x\left(t\right) = a_x + b_x t,\ t \in \left[t_1;\ t_2\right]:\ b_x \in \left[b_x^{(\min)};\ b_x^{(\max)}\right] \subset \mathbb{R}\right\} \subset \\ \subset L\left[t_1;\ t_2\right] \subset \mathbb{L}_2\left[t_1;\ t_2\right]. \tag{2}$$

For the second player, denote its offset by a_y and its trend-defining coefficient by b_y, where

$$b_y \in \left[b_y^{(\min)};\ b_y^{(\max)}\right] \text{ by } b_y^{(\max)} > b_y^{(\min)}. \tag{3}$$

Then the set of pure strategies of the second player is

$$Y = \left\{y\left(t\right) = a_y + b_y t,\ t \in \left[t_1;\ t_2\right]:\ b_y \in \left[b_y^{(\min)};\ b_y^{(\max)}\right] \subset \mathbb{R}\right\} \subset \\ \subset L\left[t_1;\ t_2\right] \subset \mathbb{L}_2\left[t_1;\ t_2\right]. \tag{4}$$

The players' payoffs in situation $\{x\left(t\right),\ y\left(t\right)\}$ are

$$K_x\left(x\left(t\right),\ y\left(t\right)\right) \text{ and } K_y\left(x\left(t\right),\ y\left(t\right)\right),$$

respectively. These payoffs are integral functionals:

$$K_x\left(x\left(t\right),\ y\left(t\right)\right) = \int\limits_{t_1}^{t_2} f\left(x\left(t\right),\ y\left(t\right)\right) dt \tag{5}$$

and

$$K_y\left(x\left(t\right),\ y\left(t\right)\right) = \int\limits_{t_1}^{t_2} g\left(x\left(t\right),\ y\left(t\right)\right) dt, \tag{6}$$

where $f\left(x\left(t\right),\ y\left(t\right)\right)$ and $g\left(x\left(t\right),\ y\left(t\right)\right)$ are algebraic functions of $x\left(t\right)$ and $y\left(t\right)$ defined everywhere on $[t_1;\ t_2]$. Therefore, the continuous noncooperative two-person game

$$\left\langle \left\{X,\ Y\right\},\ \left\{K_x\left(x\left(t\right),\ y\left(t\right)\right),\ K_y\left(x\left(t\right),\ y\left(t\right)\right)\right\}\right\rangle \tag{7}$$

is defined on product

$$X \times Y \subset L\left[t_1;\ t_2\right] \times L\left[t_1;\ t_2\right] \subset \mathbb{L}_2\left[t_1;\ t_2\right] \times \mathbb{L}_2\left[t_1;\ t_2\right] \tag{8}$$

of linear strategy functional spaces (2) and (4).

5. Acceptable solutions

Since a series of games (7) on product (8) is to be solved in practice, the only acceptable solutions are equilibrium or/and efficient situations in pure strategies. Such situations are defined similarly to those in games on finite-dimensional Euclidean subspaces [5, 10].

Definition 1. Situation $\left\{x^*\left(t\right),\ y^*\left(t\right)\right\}$ in game (7) on product (8) by conditions (1) — (6) is an equilibrium situation in pure strategies if inequalities

$$K_x\left(x\left(t\right),\ y^*\left(t\right)\right) \leqslant K_x\left(x^*\left(t\right),\ y^*\left(t\right)\right) \ \ \forall\, x\left(t\right) \in X \tag{9}$$

and

$$K_y\left(x^*\left(t\right),\ y\left(t\right)\right) \leqslant K_y\left(x^*\left(t\right),\ y^*\left(t\right)\right) \ \ \forall\, y\left(t\right) \in Y \tag{10}$$

are simultaneously true.

Definition 2. Situation $\left\{x^{**}\left(t\right),\ y^{**}\left(t\right)\right\}$ in game (7) on product (8) by conditions (1) — (6) is an efficient situation in pure strategies if both a pair of inequalities

$$\begin{gathered} K_x\left(x^{**}\left(t\right),\ y^{**}\left(t\right)\right) \leqslant K_x\left(x\left(t\right),\ y\left(t\right)\right) \ \text{ and} \\ K_y\left(x^{**}\left(t\right),\ y^{**}\left(t\right)\right) < K_y\left(x\left(t\right),\ y\left(t\right)\right) \end{gathered} \tag{11}$$

and a pair of inequalities

$$\begin{gathered} K_x\left(x^{**}\left(t\right),\ y^{**}\left(t\right)\right) < K_x\left(x\left(t\right),\ y\left(t\right)\right) \ \text{ and} \\ K_y\left(x^{**}\left(t\right),\ y^{**}\left(t\right)\right) \leqslant K_y\left(x\left(t\right),\ y\left(t\right)\right) \end{gathered} \tag{12}$$

are impossible for any $x\left(t\right) \in X$ and $y\left(t\right) \in Y$.

The continuous noncooperative two-person game can have the empty set of equilibria in pure strategies [10]. Moreover, every efficient situation can be too asymmetric, i. e. it is profitable for one player and unacceptably unprofitable for the other player. In such cases, the game does not have an acceptable solution. Then the concepts of ε-equilibrium and ε-efficiency are useful [10, 11].

Definition 3. Situation $\left\{x^{*(\varepsilon)}(t), \; y^{*(\varepsilon)}(t)\right\}$ in game (7) on product (8) by conditions (1) — (6) is an ε-equilibrium situation in pure strategies for some $\varepsilon > 0$ if inequalities

$$K_x\left(x(t), \; y^{*(\varepsilon)}(t)\right) \leqslant K_x\left(x^{*(\varepsilon)}(t), \; y^{*(\varepsilon)}(t)\right) + \varepsilon \;\; \forall x(t) \in X \tag{13}$$

and

$$K_y\left(x^{*(\varepsilon)}(t), \; y(t)\right) \leqslant K_y\left(x^{*(\varepsilon)}(t), \; y^{*(\varepsilon)}(t)\right) + \varepsilon \;\; \forall y(t) \in Y \tag{14}$$

are simultaneously true.

Definition 4. Situation $\left\{x^{**(\varepsilon)}(t), \; y^{**(\varepsilon)}(t)\right\}$ in game (7) on product (8) by conditions (1) — (6) is an ε-efficient situation in pure strategies for some $\varepsilon > 0$ if both a pair of inequalities

$$\begin{gathered} K_x\left(x^{**(\varepsilon)}(t), \; y^{**(\varepsilon)}(t)\right) + \varepsilon \leqslant K_x\left(x(t), \; y(t)\right) \;\; \text{and} \\ K_y\left(x^{**(\varepsilon)}(t), \; y^{**(\varepsilon)}(t)\right) + \varepsilon < K_y\left(x(t), \; y(t)\right) \end{gathered} \tag{15}$$

and a pair of inequalities

$$\begin{gathered} K_x\left(x^{**(\varepsilon)}(t), \; y^{**(\varepsilon)}(t)\right) + \varepsilon < K_x\left(x(t), \; y(t)\right) \;\; \text{and} \\ K_y\left(x^{**(\varepsilon)}(t), \; y^{**(\varepsilon)}(t)\right) + \varepsilon \leqslant K_y\left(x(t), \; y(t)\right) \end{gathered} \tag{16}$$

are impossible for any $x(t) \in X$ and $y(t) \in Y$.

The situations given by Definitions 1 — 4 are the acceptable solutions regardless of whether the game is finite or not. The best consequent is when a situation is simultaneously equilibrium (by Definition 1) and efficient (by Definition 2). If this is impossible, then the most preferable is an efficient situation at which the sum of players' payoffs is maximal. However, if the payoffs are unacceptably asymmetric, then the best consequent is to find such ε for which a situation is simultaneously equilibrium (by Definition 3) and efficient (by Definition 4). This approach relates to the method of solving games approximately by providing concessions [8]. Eventually, a payoff asymmetry may be smoothed by a compensation from the player whose payoff is unacceptably greater [11].

6. The finite approximation

It is obvious that, in game (7) on product (8) by conditions (1) — (6), the pure strategy of the player is determined by the trend-defining coefficient. Therefore, this game can be thought of as it is defined, instead of product (8) of linear strategy functional spaces (2) and (4), on rectangle

$$\left[b_x^{(\min)}; \; b_x^{(\max)}\right] \times \left[b_y^{(\min)}; \; b_y^{(\max)}\right] \subset \mathbb{R}^2. \tag{17}$$

This rectangle is easily sampled by using a number of equal intervals along each dimension. Denote this number by S, $S \in \mathbb{N} \backslash \{1\}$. Then

$$B_x = \left\{ b_x^{(\min)} + (s-1) \cdot \frac{b_x^{(\max)} - b_x^{(\min)}}{S} \right\}_{s=1}^{S+1} = \left\{ b_x^{(s)} \right\}_{s=1}^{S+1} \subset \left[b_x^{(\min)};\ b_x^{(\max)} \right] \quad (18)$$

and

$$B_y = \left\{ b_y^{(\min)} + (s-1) \cdot \frac{b_y^{(\max)} - b_y^{(\min)}}{S} \right\}_{s=1}^{S+1} = \left\{ b_y^{(s)} \right\}_{s=1}^{S+1} \subset \left[b_y^{(\min)};\ b_y^{(\max)} \right]. \quad (19)$$

So, rectangle (17) is substituted with grid $B_x \times B_y$. Set (18) leads to a finite set

$$X_B = \left\{ x(t) = a_x + b_x t,\ t \in [t_1;\ t_2] :\ b_x \in B_x \subset \left[b_x^{(\min)};\ b_x^{(\max)} \right] \subset \mathbb{R} \right\} =$$
$$= \left\{ x_s(t) = a_x + b_x^{(s)} t \right\}_{s=1}^{S+1} \subset X \subset L[t_1;\ t_2] \subset \mathbb{L}_2[t_1;\ t_2] \quad (20)$$

of pure strategies (linear functions of time) of the first player, where

$$x_1(t) = a_x + b_x^{(\min)} t,\ \ x_{S+1}(t) = a_x + b_x^{(\max)} t,$$

and set (19) leads to a finite set

$$Y_B = \left\{ y(t) = a_y + b_y t,\ t \in [t_1;\ t_2] :\ b_y \in B_y \subset \left[b_y^{(\min)};\ b_y^{(\max)} \right] \subset \mathbb{R} \right\} =$$
$$= \left\{ y_s(t) = a_y + b_y^{(s)} t \right\}_{s=1}^{S+1} \subset Y \subset L[t_1;\ t_2] \subset \mathbb{L}_2[t_1;\ t_2] \quad (21)$$

of pure strategies (linear functions of time) of the second player, where

$$y_1(t) = a_y + b_y^{(\min)} t,\ \ y_{S+1}(t) = a_y + b_y^{(\max)} t.$$

Subsequently, game (7) on product (8) by conditions (1) — (6) is substituted with a finite game

$$\langle \{X_B,\ Y_B\},\ \{K_x(x(t),\ y(t)),\ K_y(x(t),\ y(t))\} \rangle$$
$$\text{by } x(t) \in X_B \text{ and } y(t) \in Y_B \quad (22)$$

defined on product

$$X_B \times Y_B \subset X \times Y \subset L[t_1;\ t_2] \times L[t_1;\ t_2] \subset \mathbb{L}_2[t_1;\ t_2] \times \mathbb{L}_2[t_1;\ t_2] \quad (23)$$

of linear strategy functional subspaces (20) and (21). In fact, game (22) is a bimatrix $(S+1) \times (S+1)$-game.

To perform an appropriate approximation via the sampling, number S is selected so that none of S^2 rectangles

$$\left[b_x^{(i)};\ b_x^{(i+1)} \right] \times \left[b_y^{(j)};\ b_y^{(j+1)} \right] \text{ by } i = \overline{1,\ S} \text{ and } j = \overline{1,\ S} \quad (24)$$

would contain significant specificities of payoff functionals (5) and (6). In fact, such specificities are extremals of these functionals.

Theorem 1. *In game (7) on product (8) by conditions (1) — (6), the player's payoff functional achieves its maximal and minimal values on any closed subset of rectangle (17) of the trend-defining coefficients.*

Proof. Both $f(x(t), y(t))$ and $g(x(t), y(t))$ are algebraic functions of linear functions $x(t)$ and $y(t)$ defined everywhere on $[t_1;\ t_2]$. Therefore, both integrals in functionals (5) and (6) achieve some maximal and minimal values on any closed subset of rectangle (17) of the trend-defining coefficients. □

Thus, Theorem 1 allows controlling extremals of payoff functionals (5) and (6) by the trend-defining coefficient. Moreover, Theorem 1 is easily expanded on closed rectangles (24) for any number S. Consequently, if inequalities

$$\max_{\substack{b_x \in \left[b_x^{(i)};\ b_x^{(i+1)}\right], \\ b_y \in \left[b_y^{(j)};\ b_y^{(j+1)}\right]}} K_x(x(t), y(t)) - \min_{\substack{b_x \in \left[b_x^{(i)};\ b_x^{(i+1)}\right], \\ b_y \in \left[b_y^{(j)};\ b_y^{(j+1)}\right]}} K_x(x(t), y(t)) =$$
$$= \max_{\substack{b_x \in \left[b_x^{(i)};\ b_x^{(i+1)}\right], \\ b_y \in \left[b_y^{(j)};\ b_y^{(j+1)}\right]}} \int_{t_1}^{t_2} f(x(t), y(t))\, dt - \min_{\substack{b_x \in \left[b_x^{(i)};\ b_x^{(i+1)}\right], \\ b_y \in \left[b_y^{(j)};\ b_y^{(j+1)}\right]}} \int_{t_1}^{t_2} f(x(t), y(t))\, dt \leqslant \mu$$
$$\forall\, i = \overline{1,\ S} \text{ and } \forall\, j = \overline{1,\ S} \tag{25}$$

and

$$\max_{\substack{b_x \in \left[b_x^{(i)};\ b_x^{(i+1)}\right], \\ b_y \in \left[b_y^{(j)};\ b_y^{(j+1)}\right]}} K_y(x(t), y(t)) - \min_{\substack{b_x \in \left[b_x^{(i)};\ b_x^{(i+1)}\right], \\ b_y \in \left[b_y^{(j)};\ b_y^{(j+1)}\right]}} K_y(x(t), y(t)) =$$
$$= \max_{\substack{b_x \in \left[b_x^{(i)};\ b_x^{(i+1)}\right], \\ b_y \in \left[b_y^{(j)};\ b_y^{(j+1)}\right]}} \int_{t_1}^{t_2} g(x(t), y(t))\, dt - \min_{\substack{b_x \in \left[b_x^{(i)};\ b_x^{(i+1)}\right], \\ b_y \in \left[b_y^{(j)};\ b_y^{(j+1)}\right]}} \int_{t_1}^{t_2} g(x(t), y(t))\, dt \leqslant \mu$$
$$\forall\, i = \overline{1,\ S} \text{ and } \forall\, j = \overline{1,\ S} \tag{26}$$

are simultaneously true for some sufficiently small $\mu > 0$, then those μ-variations can be ignored. Thus, for the properly selected S and μ, game (7) on product (8) by conditions (1) — (6) can be approximated by finite game (22). The quality of the approximation can be comprehended by the following assertions.

Theorem 2. *If $\{x^*(t), y^*(t)\}$ is an equilibrium in game (7) on product (8) by conditions (1) — (6), where functionals (5) and (6) are continuous, conditions (25) and (26) hold for some S and μ,*

$$x^*(t) = a_x + b_x^* t \quad \text{by} \quad b_x^* \in \left[b_x^{(h)};\ b_x^{(h+1)}\right] \quad \text{and}$$
$$y^*(t) = a_y + b_y^* t \quad \text{by} \quad b_y^* \in \left[b_y^{(k)};\ b_y^{(k+1)}\right]$$
$$\text{for } h \in \left\{\overline{1,\ S}\right\},\ k \in \left\{\overline{1,\ S}\right\}, \tag{27}$$

then every situation $\left\{x^{*(\varepsilon)}(t),\ y^{*(\varepsilon)}(t)\right\}$ *for which*

$$x^{*(\varepsilon)}(t) = a_x + b_x^{*(\varepsilon)} t \quad by \quad b_x^{*(\varepsilon)} \in \left[b_x^{(h)};\ b_x^{(h+1)}\right] \quad and$$
$$y^{*(\varepsilon)}(t) = a_y + b_y^{*(\varepsilon)} t \quad by \quad b_y^{*(\varepsilon)} \in \left[b_y^{(k)};\ b_y^{(k+1)}\right]$$
$$for\ \ h \in \left\{\overline{1,\ S}\right\},\ \ k \in \left\{\overline{1,\ S}\right\}, \tag{28}$$

is an ε*-equilibrium for some* $\varepsilon > 0$*. As number* S *is increased, the value of* ε *can be made smaller.*

Proof. Whichever integer S and the corresponding μ are, the value of ε always can be chosen such that inequalities (13) and (14) be true for every situation composed of strategies (28) by (27). It is obvious that, owing to the continuity of the functionals, increasing number S allows decreasing the value of μ, which provides ε-equilibria to be closer to the equilibrium composed of strategies (27). □

Theorem 3. *If* $\{x^{**}(t),\ y^{**}(t)\}$ *is an efficient situation in game (7) on product (8) by conditions (1) — (6), where functionals (5) and (6) are continuous, conditions (25) and (26) hold for some* S *and* μ*,*

$$x^{**}(t) = a_x + b_x^{**} t \quad by \quad b_x^{**} \in \left[b_x^{(h)};\ b_x^{(h+1)}\right] \quad and$$
$$y^{**}(t) = a_y + b_y^{**} t \quad by \quad b_y^{**} \in \left[b_y^{(k)};\ b_y^{(k+1)}\right]$$
$$for\ \ h \in \left\{\overline{1,\ S}\right\},\ \ k \in \left\{\overline{1,\ S}\right\}, \tag{29}$$

then every situation $\left\{x^{**(\varepsilon)}(t),\ y^{**(\varepsilon)}(t)\right\}$ *for which*

$$x^{**(\varepsilon)}(t) = a_x + b_x^{**(\varepsilon)} t \quad by \quad b_x^{**(\varepsilon)} \in \left[b_x^{(h)};\ b_x^{(h+1)}\right] \quad and$$
$$y^{**(\varepsilon)}(t) = a_y + b_y^{**(\varepsilon)} t \quad by \quad b_y^{**(\varepsilon)} \in \left[b_y^{(k)};\ b_y^{(k+1)}\right]$$
$$for\ \ h \in \left\{\overline{1,\ S}\right\},\ \ k \in \left\{\overline{1,\ S}\right\}, \tag{30}$$

is an ε*-efficient situation for some* $\varepsilon > 0$*. As number* S *is increased, the value of* ε *can be made smaller.*

Proof. Whichever integer S and the corresponding μ are, value ε always can be chosen such that inequalities (15) and (16) be true for every situation composed of strategies (30) by (29). It is obvious that, owing to the continuity of the functionals, increasing number S allows decreasing the value of μ, which provides ε-efficient situations to be closer to the efficient situation composed of strategies (29). □

Hence, the finite approximation should start from some integer S, for which a bimatrix $(S+1)\times(S+1)$-game (22) is built and solved. Then this integer is gradually increased (although, the increment is not ascertained for general case), and new, bigger, bimatrix games are solved. The process can be stopped if the acceptable solution (whether it is an equilibrium, efficient, ε-equilibrium, or ε-efficient situation) by the last iteration does not differ much from the acceptable solutions (of the same type) by a few preceding iterations. Thus, if

$$\left\{x^{<l>*}(t),\ y^{<l>*}(t)\right\} = \left\{a_x + b_x^{<l>*}t,\ a_y + b_y^{<l>*}t\right\} \in X_B \times Y_B \subset X \times Y \quad (31)$$

is an acceptable solution at the l-th iteration, then the conditions of the sufficient closeness to the solutions at the preceding and succeeding iterations are as follows:

$$\sqrt{\int_{t_1}^{t_2} \left(x^{<l-1>*}(t) - x^{<l>*}(t)\right)^2 dt} \geqslant \sqrt{\int_{t_1}^{t_2} \left(x^{<l>*}(t) - x^{<l+1>*}(t)\right)^2 dt} \quad \text{and}$$

$$\sqrt{\int_{t_1}^{t_2} \left(y^{<l-1>*}(t) - y^{<l>*}(t)\right)^2 dt} \geqslant \sqrt{\int_{t_1}^{t_2} \left(y^{<l>*}(t) - y^{<l+1>*}(t)\right)^2 dt} \quad (32)$$

and

$$\max_{t\in[t_1;\, t_2]} \left|x^{<l-1>*}(t) - x^{<l>*}(t)\right| \geqslant \max_{t\in[t_1;\, t_2]} \left|x^{<l>*}(t) - x^{<l+1>*}(t)\right| \quad \text{and}$$

$$\max_{t\in[t_1;\, t_2]} \left|y^{<l-1>*}(t) - y^{<l>*}(t)\right| \geqslant \max_{t\in[t_1;\, t_2]} \left|y^{<l>*}(t) - y^{<l+1>*}(t)\right| \quad (33)$$

by $l = 2,\ 3,\ 4,\ \ldots$

Theorem 4. *Conditions (32) and (33) of the sufficient closeness for game (7) on product (8) by conditions (1) — (6) are expressed as*

$$\left|b_x^{<l-1>*} - b_x^{<l>*}\right| \geqslant \left|b_x^{<l>*} - b_x^{<l+1>*}\right| \quad by \quad l = 2,\ 3,\ 4,\ \ldots \quad (34)$$

and

$$\left|b_y^{<l-1>*} - b_y^{<l>*}\right| \geqslant \left|b_y^{<l>*} - b_y^{<l+1>*}\right| \quad by \quad l = 2,\ 3,\ 4,\ \ldots \quad (35)$$

Proof. Due to that

$$\sqrt{\int_{t_1}^{t_2} \left(x^{<l-1>*}(t) - x^{<l>*}(t)\right)^2 dt} = \sqrt{\int_{t_1}^{t_2} \left(a_x + b_x^{<l-1>*}t - a_x - b_x^{<l>*}t\right)^2 dt} =$$

$$= \sqrt{\int_{t_1}^{t_2} \left(b_x^{<l-1>*} - b_x^{<l>*}\right)^2 t^2 dt} = \sqrt{\left(b_x^{<l-1>*} - b_x^{<l>*}\right)^2 \left(\frac{t_2^3}{3} - \frac{t_1^3}{3}\right)} =$$

$$= \left|b_x^{<l-1>*} - b_x^{<l>*}\right| \sqrt{\frac{t_2^3 - t_1^3}{3}}$$

and

$$\max_{t\in[t_1;\, t_2]} \left|x^{<l-1>*}(t) - x^{<l>*}(t)\right| = \max_{t\in[t_1;\, t_2]} \left|\left(b_x^{<l-1>*} - b_x^{<l>*}\right) t\right| =$$

$$= \left| b_x^{<l-1>*} - b_x^{<l>*} \right| t_2$$

(where time is presumed to be nonnegative), inequalities (32) and (33) are simplified explicitly:

$$\left| b_x^{<l-1>*} - b_x^{<l>*} \right| \sqrt{\frac{t_2^3 - t_1^3}{3}} \geqslant \left| b_x^{<l>*} - b_x^{<l+1>*} \right| \sqrt{\frac{t_2^3 - t_1^3}{3}} \text{ and}$$
$$\left| b_y^{<l-1>*} - b_y^{<l>*} \right| \sqrt{\frac{t_2^3 - t_1^3}{3}} \geqslant \left| b_y^{<l>*} - b_y^{<l+1>*} \right| \sqrt{\frac{t_2^3 - t_1^3}{3}}$$

and

$$\left| b_x^{<l-1>*} - b_x^{<l>*} \right| t_2 \geqslant \left| b_x^{<l>*} - b_x^{<l+1>*} \right| t_2 \text{ and}$$
$$\left| b_y^{<l-1>*} - b_y^{<l>*} \right| t_2 \geqslant \left| b_y^{<l>*} - b_y^{<l+1>*} \right| t_2,$$

whence they are expressed as (34) and (35), respectively. □

If inequalities (34) and (35) hold for at least three iterations, the approximation procedure can be stopped. Clearly, the closeness strengthens if inequalities (34) and (35) hold strictly. However, inequalities (34) and (35) may not hold for a wide range of iterations, so it is better to require that both polylines

$$\lambda_x(l) = \left| b_x^{<l>*} - b_x^{<l+1>*} \right| \text{ by } l = 1,\ 2,\ 3,\ \ldots \tag{36}$$

and

$$\lambda_y(l) = \left| b_y^{<l>*} - b_y^{<l+1>*} \right| \text{ by } l = 1,\ 2,\ 3,\ \ldots \tag{37}$$

be decreasing on average. Herein, term "on average" implies that, in the case when inequalities (34) and (35) do not hold, polylines (36) and (37) are smoothed (approximated) with the respective polynomials of degree 2.

7. Exemplification

To exemplify the method of the game finite approximation, consider a case in which $t \in [1;\ 30]$, the set of pure strategies of the first player is

$$X = \{x(t) = 100 + b_x t,\ t \in [1;\ 30] :\ b_x \in [-0.4;\ 0.4] \subset \mathbb{R}\} \subset$$
$$\subset L[1;\ 30] \subset \mathbb{L}_2[1;\ 30], \tag{38}$$

and the set of pure strategies of the second player is

$$Y = \{y(t) = 120 + b_y t,\ t \in [1;\ 30] :\ b_y \in [-0.6;\ 0.6] \subset \mathbb{R}\} \subset$$
$$\subset L[1;\ 30] \subset \mathbb{L}_2[1;\ 30]. \tag{39}$$

The payoff functionals are

$$K_x(x(t),\ y(t)) = \int_1^{30} 10000 \cdot \frac{5x^2(t) + x(t) - x(t)\,y(t) - y^2(t)}{x^3(t) + x^2(t) + x(t) - x(t)\,y(t) - y^2(t)} dt \tag{40}$$

and

$$K_y\left(x\left(t\right),\ y\left(t\right)\right) = \int_{1}^{30} \left(y\left(t\right) - 1.2x\left(t\right)\right)^2 dt. \tag{41}$$

Consequently, this game can be thought of as it is defined on rectangle (17):

$$[-0.4;\ 0.4] \times [-0.6;\ 0.6] \subset \mathbb{R}^2. \tag{42}$$

It is easy to show that functional (40) is continuous (Figure 1). The continuity of functional (41) is quite clear (Figure 2). Therefore, Theorem 2 and Theorem 3 will ensure fast approximation here. At $S = 5$ the respective bimatrix 6×6-game has a single equilibrium and two efficient situations. By increasing the number of intervals

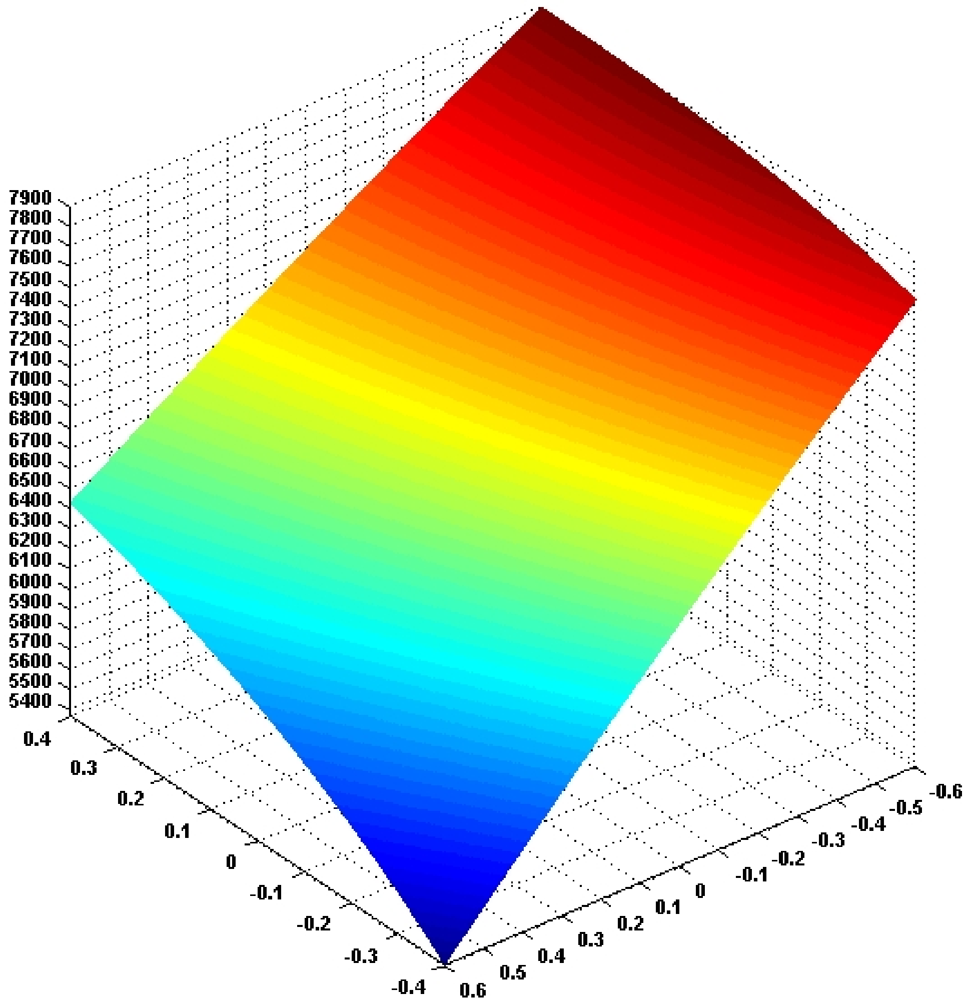

Figure 1: The first player's payoff functional (40) shown on rectangle (42)

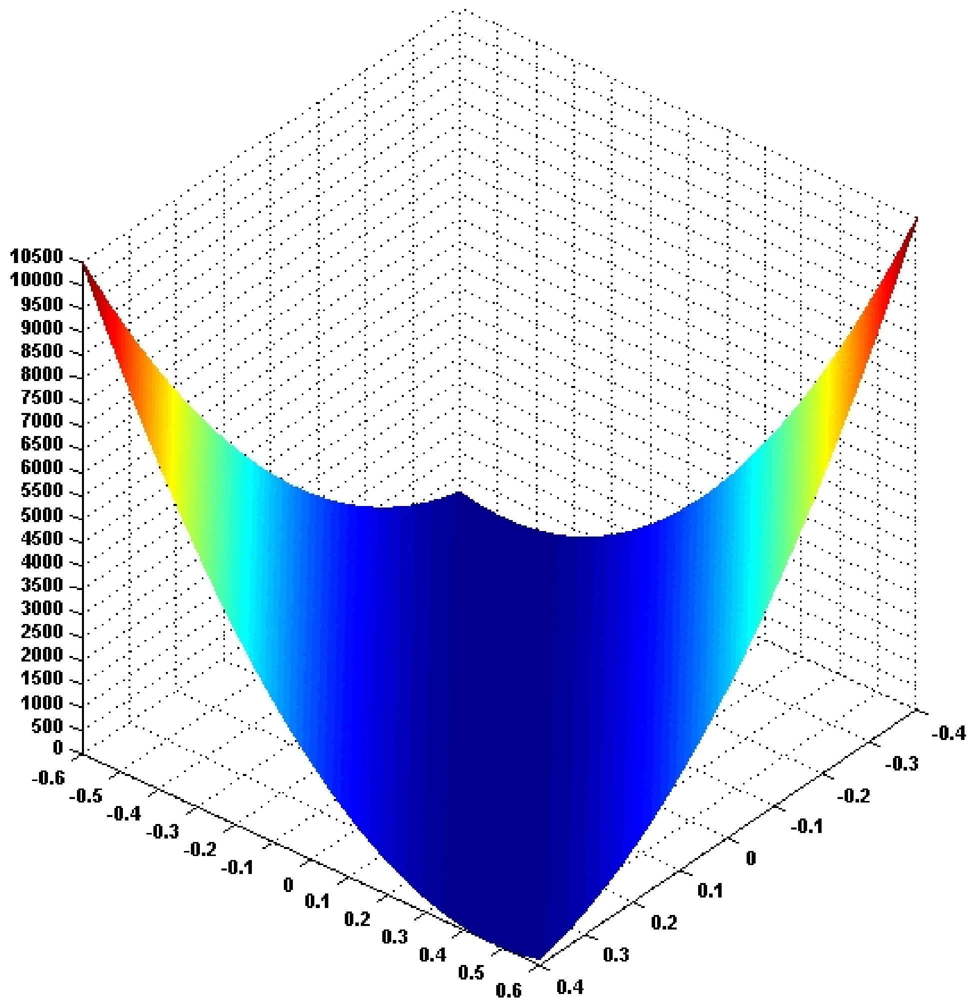

Figure 2: The second player's payoff functional (41) shown on rectangle (42)

with a step of 5 up to 100, a single equilibrium is still found, but the number of efficient situations grows. One of those efficient situations is equilibrium (by Definition 1). In such a situation, the equilibrium-and-efficient strategies of the first player become "stable" as S increases (Figure 3). Eventually,

$$x^{<20>*}(t) = 100 + 0.344t,$$

whereas the equilibrium-and-efficient strategy of the second player remains the same for all $S = 5,\ 10,\ 15,\ \ldots,\ 100$ (Figure 4). So, condition (35) of the sufficient closeness of the second player's strategies holds trivially. After all, the first player's polyline by (36) decreases on average (Figure 5). This means that situation

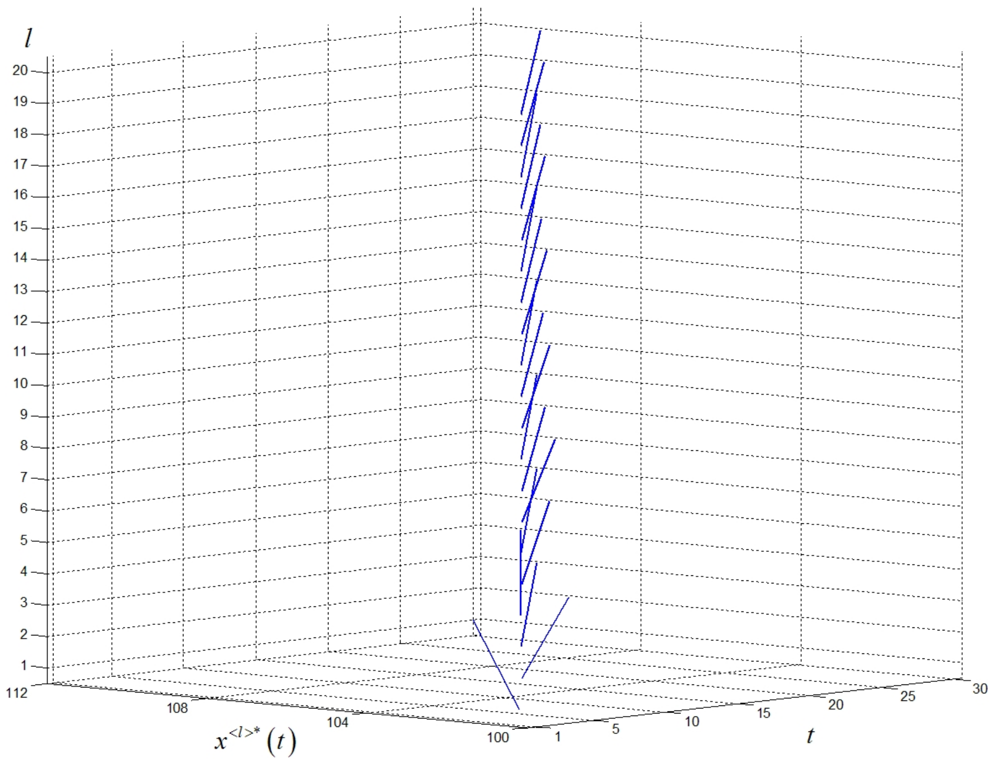

Figure 3: The series of 20 equilibrium-and-efficient strategies of the first player

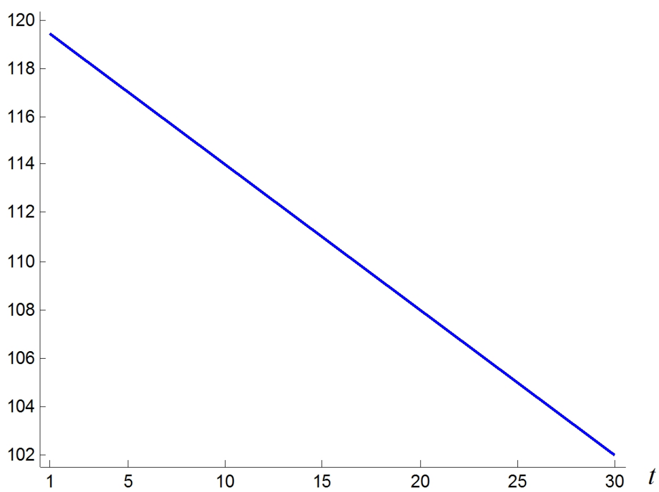

Figure 4: The second player's unvarying equilibrium-and-efficient strategy $y^{<l>*}(t) = 120 - 0.6t \quad (l = \overline{1,\ 20})$

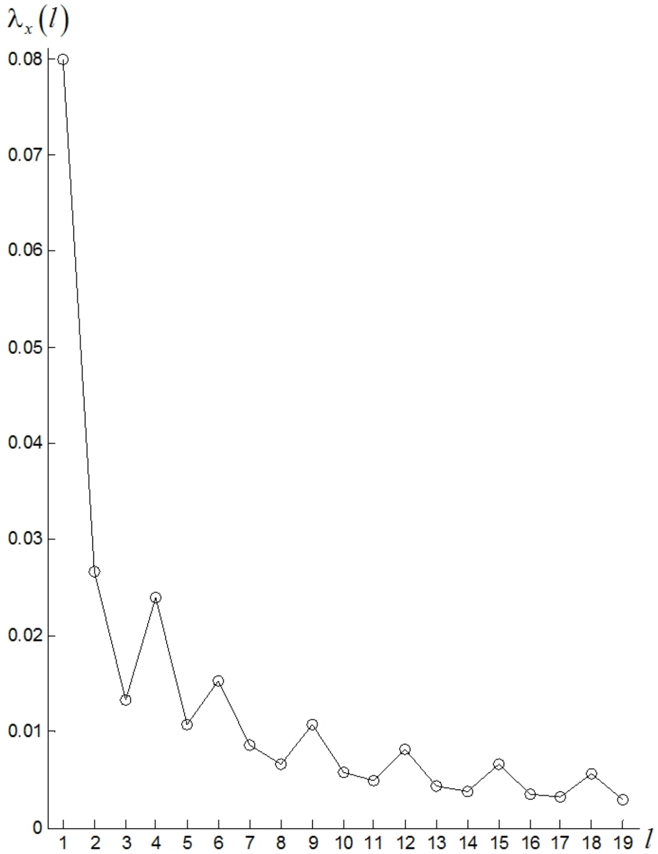

Figure 5: The first player's polyline from (36), which decreases on average

$$\left\{x^{<20>*}(t)\,,\ y^{<20>*}(t)\right\} = \{100 + 0.344t,\ 120 - 0.6t\}$$

is the solution of the corresponding bimatrix 101 × 101-game, which is the single acceptable approximate solution to the initial game with (38) — (41).

8. Discussion

Continuous games are approximated to finite games not just for the sake of simplicity itself. The matter is the finite approximation makes solutions tractable so that they can be easily implemented and practiced. So, the presented method of finite approximation specifies and, what is more important, establishes the applicability of continuous noncooperative two-person games on a product of linear strategy functional spaces. Mainly, it concerns modeling economic interaction processes, where the player can use a continuum of short-term time-varying strategies.

The presented method is quite significant for avoiding too complicated solutions resulting from game continuities and, moreover, functional spaces of pure strategies. This is similar to preventing Einstellung effect in modeling [4]. The transfer from a functional space product to a real-valued rectangle with subsequently sampling it into a grid herein "deeinstellungizes" the continuous noncooperative two-person game.

9. Conclusion

For solving continuous noncooperative two-person games on a product of linear strategy functional spaces, a method of their finite approximation is presented, which is based on sampling the linear strategy functional spaces. The sets (i. e., the spaces) of the players' pure strategies are sampled uniformly so that the resulting finite game is a bimatrix game whose payoff matrices are square. The approximation procedure starts with not a great number of intervals. Then this number is gradually increased, and new, bigger, bimatrix games are solved until an acceptable solution of the bimatrix game becomes sufficiently close to the same-type solutions at the preceding iterations. The closeness is expressed in terms of the respective functional spaces, which is simplified by Theorem 4, giving just the absolute difference between the trend-defining coefficients of the strategies from the neighboring solutions. These distances should be decreasing once they are smoothed with respective polynomials of degree 2.

A question of the game finite approximation for cases of nonlinear strategy spaces (when, say, the player's strategy space is of parabolas or cubic polynomials) is believed to be answered in the similar manner. Nevertheless, some peculiarities concerning the continuity of the payoff functionals may weaken the impact of Theorem 2 and Theorem 3. Despite this, the game finite approximation will definitely have an expansion in order not to admit the above-mentioned Einstellung effect in modeling economic interaction processes, where players use short-term time-varying strategies of various types.

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DOI: 10.7862/rf.2020.9

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Received 27.04.2020 *Accepted 18.07.2020*