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Difference Sequence Spaces Defined by Musielak-Orlicz Function

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Abstract: The purpose of this paper is to introduce sequence spaces $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p), \|., .., .\|\ ]$ and $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p), \|., .., .\|\ ]_{\theta}$. We also examine some topological properties and prove some inclusion relations between these spaces.

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1. Introduction and Preliminaries

The notion of the difference sequence space was introduced by Kızmaz [1]. It was further generalized by Et and Çolak [2] as follows: $Z(\Delta^{\mu}) = \{x = (x_k) \in \omega : (\Delta^{\mu} x_k) \in z\}$ for $z = \ell_{\infty},\ c$ and $c_{\circ}$, where μ is a non-negative integer and

$$\Delta^{\mu} x_k = \Delta^{\mu-1} x_k - \Delta^{\mu-1} x_{k+1},\ \Delta^{\circ} x_k = x_k \text{ for all } k \in \mathbb{N}$$

or equivalent to the following binomial representation:

$$\Delta^{\mu} x_k = \sum_{v=0}^{\mu} (-1)^v \binom{\mu}{v} x_{k+v}.$$

These sequence spaces were generalized by Et and Basaşir [3] taking $z = \ell_{\infty}(p)$, $c(p)$ and $c_{\circ}(p)$. Dutta [4] introduced the following difference sequence spaces using a new difference operator:

$$Z(\Delta_{(\eta)}) = \left\{x = (x_k) \in \omega : \Delta_{(\eta)} x \in z\right\} \text{ for } z = \ell_{\infty},\ c \text{ and } c_{\circ},$$

where $\Delta_{(\eta)}x = (\Delta_{(\eta)}x_k) = (x_k - x_{k-\eta})$ for all $k, \eta \in \mathbb{N}$.
In [5], Dutta introduced the sequence spaces $\bar{c}(\|.,.\|,\ \Delta^{\mu}_{(\eta)}, p)$, $\bar{c_\circ}(\|.,.\|,\ \Delta^{\mu}_{(\eta)}, p)$, $\ell_\infty(\|.,.\|,\ \Delta^{\mu}_{(\eta)}, p)$, $m(\|.,.\|,\ \Delta^{\mu}_{(\eta)}, p)$ and $m_\circ(\|.,.\|,\ \Delta^{\mu}_{(\eta)}, p)$, where $\eta,\ \mu \in \mathbb{N}$ and $\Delta^{\mu}_{(\eta)}x_k = (\Delta^{\mu}_{(\eta)}x_k) = (\Delta^{\mu-1}_{(\eta)}x_k - \Delta^{\mu-1}_{(\eta)}x_{k-\eta})$ and $\Delta^{\circ}_{(\eta)}x_k = x_k$ for all $k, \eta \in \mathbb{N}$, which is equivalent to the following binomial representation:

$$\Delta^{\mu}_{(\eta)}x = \sum_{v=0}^{\mu}(-1)^v \binom{\mu}{v} x_{k-\eta v}.$$

The difference sequence space have been studied by authors ([6], [7], [8], [9], [10], [11], [12], [13], [14], [15]) and references therein. Başar and Altay [16] introduced the generalized difference matrix $B = (b_{mk})$ for all $k, m \in \mathbb{N}$, which is a generalization of $\Delta_{(1)}$-difference operator, by

$$b_{mk} = \begin{cases} r, & k = m \\ s, & k = m-1 \\ 0, & (k > m) \text{ or } (0 \le k < m-1). \end{cases}$$

Başarir and Kayikçi [17] defined the matrix $B^{\mu}(b^{\mu}_{mk})$ which reduced the difference matrix $\Delta^{\mu}_{(1)}$ incase $r = 1,\ s = -1$. The generalized B^{μ}-difference operator is equivalent to the following binomial representation:

$$B^{\mu}x = B^{\mu}(x_k) = \sum_{v=0}^{\mu} \binom{\mu}{v} r^{\mu-v} s^v x_{k-v}.$$

Let $\wedge = (\wedge_k)$ be a sequence of non-zero scalars. Then, for a sequence space E, the multiplier sequence space $E_\wedge$, associated with the multiplier sequence $\wedge$, is defined as

$$E_\wedge = \{x = (x_k) \in \omega : (\wedge_k x_k) \in E\}.$$

An Orlicz function M is a function, which is continuous non-decreasing and convex with $M(0) = 0,\ M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.
Linderstrauss and Tzafriri [18] used the idea of Orlicz function to define the following sequence space

$$\ell_M = \left\{ x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \right\}$$

which is called an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}.$$

It is shown in [18] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \ge 1)$. The Δ_2-condition is equivalent to $M(Lx) \le KLM(x)$ for all values of

$x \geq 0$ and for $L > 1$. An Orlicz function M can always be represented in the following interval form

$$M(x) = \int_0^x \eta(t)dt,$$

where η is known as the kernel of M, is right differentiable for $t \geq 0$, $\eta(0) = 0$, $\eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.
A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a Musielak-Orlicz function see ([19], [20]). A sequence $\mathcal{N} = (N_k)$ defined by

$$N_k(v) = \sup\left\{|v|u - M_k(u) : u \geq 0\right\}, \; k = 1, 2, ...$$

is called the complimentary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows:

$$t_{\mathcal{M}} = \{x \in \omega : I_M(cx) < \infty \text{ for some } c > 0\},$$

$$h_{\mathcal{M}} = \{x \in \omega : I_M(cx) < \infty \text{ for all } c > 0\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), \; x = (x_k) \in t_M.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$\|x\| = \inf\left\{k > 0 : I_M(x/k) \leq 1\right\}$$

or equipped with the Orlicz norm

$$\|x\|^{\circ} = \inf\left\{\frac{1}{k}\big(1 + I_M(kx)\big) : k > 0\right\}.$$

The concept of 2-normed spaces was initially developed by Gähler [21] in the mid of 1960's, while that of n -normed spaces one can see in Misiak [22]. Since then, many others have studied this concept and obtained various results, see Gunawan ([23], [24]) and Gunawan and Mashadi [25]. Let $n \in \mathbb{N}$ and X be linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is the field of real or complex numbers of dimension d, where $d \geq n \geq 2$.
A real valued function $\|., ..., .\|$ on X^n satisfying the following four conditions:
(1) $\|x_1, x_2, .., x_n\| = 0$ if and only if $x_1, x_2, x_3, ...x_n$ are linearly dependent in X;
(2) $\|x_1, x_2, .., x_n\|$ is invariant under permutation ;
(3) $\|\alpha x_1, x_2, .., x_n\| = |\alpha| \|x_1, x_2, .., x_n\|$ for any $\alpha \in \mathbb{K}$ and
(4) $\|x + x', x_2, .., x_n\| \leq \|x, x_2, .., x_n\| + \|x', x_2, .., x_n\|$
is called an n-norm on X and the pair $(X, \|., .., .\|)$ is called a n-normed space over the field $\mathbb{K}$. For example, we may take $X = \mathbb{R}^n$ being equipped with the Euclidean

n-norm $\|x_1, x_2, .., x_n\|_E$ as the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, .., x_n$ which may be given explicitly by the formula

$$\|x_1, x_2, .., x_n\|_E = |det(x_{ij})|;$$

where $x_i = (x_{i1}, x_{i2}, x_{i3}, ..., x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, 3, ..., n$ and $\|.\|_E$ denotes the Euclidean norm. Let $(X, \|., .., .\|)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, ..., a_n\}$ be linearly independent set in X. Then the following function $\|., ..., .\|_\infty$ on X^{n-1} defined by

$$\|x_1, x_2, .., x_n\|_\infty = \max\big\{\|x_1, x_2, .., x_{n-1}, a_i\| : i = 1, 2, ..., n\big\}$$

defines an $(n-1)$ norm on X with respect to $\{a_1, a_2, ..., a_n\}$.
A sequence (x_k) in a n-normed space $(X, \|., ..., .\|)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} \|x_k - L, z_1, .., z_{n-1}\| = 0, \text{ for every } z_1, ..., z_{n-1} \in X.$$

A sequence (x_k) in a normed space $(X, \|., ..., .\|)$ is said to be Cauchy if

$$\lim_{\substack{k\to\infty \\ p\to\infty}} \|x_k - x_p, z_1, .., z_{n-1}\| = 0, \text{ for every } z_1, ..., z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$ then X is said to be complete with respect to the n -norm. Any complete n-normed space is said to be n-Banach space.

A sequence $x = (x_k) \in l_\infty$, the space of bounded sequence is said to be almost convergent to s if $\lim_{k\to\infty} t_{km}(x) = s$ uniformly in m where $t_{km}(x) = \frac{1}{k+1}\sum_{i=0}^{k} x_{m+i}$ (see [26]).
By a lacunary sequence $\theta = (k_r)$, $r = 0, 1, 2, \cdots$, where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$. The intervals determined by θ are denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. The space N_θ of lacunary strongly convergent sequences was defined by Freedman et al. [27] as follows:

$$N_\theta = \Big\{x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_k - L| = 0 \text{ for some } L\Big\}.$$

By $[\hat{w}]_\theta$, we denote the set of all lacunary $[\hat{w}]$-convergent sequences and we write $[\hat{w}]_\theta - \lim x = s$, for $x \in [\hat{w}]_\theta$.
Let $(X, ||\cdot, \cdots, \cdot||)$ be a n-normed space and $w(n-X)$ denotes the space of X-valued sequences. Let $p = (p_k)$ be any bounded sequence of positive real numbers and $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. In this paper, we define the following sequence spaces

$[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ] =$

$= \Big\{ x = (x_k) \in w(n-X) : \frac{1}{n+1} \sum_{k=0}^{n} M_k \Big(|| \frac{t_{km}(B^{\mu}_{\Lambda}(x-s))}{\rho}, z_1, \cdots, z_{n-1} || \Big)^{p_k} \to 0$

as $n \to \infty$, uniformly in m, for some $s \Big\}$

and

$[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]_{\theta} =$

$\Big\{ x = (x_k) \in w(n-X) : \sup_m \frac{1}{h_r} \sum_{k \in I_r} M_k \Big(|| \frac{t_{km}(B^{\mu}_{\Lambda}(x-s))}{\rho}, z_1, \cdots, z_{n-1} || \Big)^{p_k} \to 0$

as $r \to \infty$, for some $s \Big\}$.

The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = H$, $K = \max(1, 2^{H-1})$ then

$$|a_k + b_k|^{p_k} \leq K\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1.1}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

The main goal of the present paper is to introduce $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p), \|., .., .\|\ ]$ and $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p), \|., .., .\|\ ]_{\theta}$. We also examine some topological properties and prove some inclusion relation between these spaces.

2. Main Results

Theorem 2.1. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers. Then $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]$ and $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]_{\theta}$ are linear spaces over the field of complex number $\mathbb{C}$.*

Proof. Let $x, y \in [\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]$ and $\alpha, \beta \in \mathbb{C}$. In order to prove the result we need to find some $\rho_3 > 0$ such that

$$\frac{1}{n+1} \sum_{k=0}^{n} M_k \Big(|| \frac{t_{km}\big(B^{\mu}_{\Lambda}(\alpha x + \beta y) - s\big)}{\rho_3}, z_1, \cdots, z_{n-1} || \Big)^{p_k} \to 0 \text{ as } n \to \infty, \text{ uniformly in}$$

m, for some s.

Since $x, y \in [\ \hat{w}(\mathcal{M}, p, \|., \cdots, .\|)\ ]$ there exist positive numbers $\rho_1,\ \rho_2$ such that

$$\frac{1}{n+1} \sum_{k=0}^{n} M_k \Big(|| \frac{t_{km}(B^{\mu}_{\Lambda}(x-s))}{\rho_1}, z_1, \cdots, z_{n-1} || \Big)^{p_k} \to 0 \text{ as } n \to \infty, \text{ uniformly in } m, \text{ for}$$

some s

and

$$\frac{1}{n+1} \sum_{k=0}^{n} M_k \Big(|| \frac{t_{km}(B^{\mu}_{\Lambda}(y-s))}{\rho_2}, z_1, \cdots, z_{n-1} || \Big)^{p_k} \to 0 \text{ as } n \to \infty, \text{ uniformly in } m, \text{ for}$$

some s.
Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\mathcal{M}$ is non-decreasing and convex

$$\begin{aligned}
\frac{1}{n+1} & \sum_{k=0}^{n} M_k\Big(||\frac{t_{km}\big(B^{\mu}_{\Lambda}(\alpha x+\beta y)-s\big)}{\rho_3}, z_1, \cdots, z_{n-1}||\Big)^{p_k} \\
\leq & \frac{1}{n+1}\sum_{k=0}^{n} M_k\Big(||\frac{t_{km}(B^{\mu}_{\Lambda}(\alpha x-s))}{\rho_3}, z_1, \cdots, z_{n-1}|| \\
& +||\frac{t_{km}(B^{\mu}_{\Lambda}(\beta y-s))}{\rho_3}, z_1, \cdots, z_{n-1}||\Big)^{p_k} \\
\leq & \frac{1}{n+1}\sum_{k=0}^{n} M_k\Big(||\frac{t_{km}(B^{\mu}_{\Lambda}(x-s))}{\rho_1}, z_1, \cdots, z_{n-1}|| \\
& +||\frac{t_{km}(B^{\mu}_{\Lambda}(y-s))}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)^{p_k} \\
\leq & K\frac{1}{n+1}\sum_{k=0}^{n} M_k\Big(||\frac{t_{km}(B^{\mu}_{\Lambda}(x-s))}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)^{p_k} \\
+ & K\frac{1}{n+1}\sum_{k=0}^{n} M_k\Big(||\frac{t_{km}(B^{\mu}_{\Lambda}(y-s))}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)^{p_k} \\
& \to 0 \text{ as } n\to\infty, \text{ uniformly in } m, \text{ for some } s.
\end{aligned}$$

Thus $\alpha x + \beta y \in [\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|.,..,.\|)\ ]$. This proves that $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|.,..,.\|)\ ]$ is a linear space. Similarly, we can prove that $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|.,..,.\|)\ ]_{\theta}$ is a linear space. □

Theorem 2.2. *Let $\theta = (k_r)$ be a lacunary sequence with* $\liminf q_r > 1$. *Then* $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|.,..,.\|)\ ] \subset [\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|.,..,.\|)\ ]_{\theta}$ *and* $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|.,..,.\|)\ ] - \lim x = [\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|.,..,.\|)\ ]_{\theta} - \lim x$.

Proof. Let $\liminf q_r > 1$. Then there exists $\delta > 0$ such that $q_r > 1+\delta$ and hence

$$\frac{h_r}{k_r} = 1 - \frac{k_{r-1}}{k_r} > 1 - \frac{1}{1+\delta} = \frac{\delta}{1+\delta}.$$

Therefore,

$$\begin{aligned}
\tfrac{1}{k_r} & \sum_{i=1}^{k_r} M_k\Big(||\frac{t_{im}(B^{\mu}_{\Lambda}(x-s))}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} \\
\geq & \frac{1}{k_r}\sum_{i\in I_r} M_k\Big(||\frac{t_{im}(B^{\mu}_{\Lambda}(x-s))}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} \\
\geq & \frac{\delta}{1+\delta}\frac{1}{h_r}\sum_{i\in I_r} M_k\Big(||\frac{t_{im}(B^{\mu}_{\Lambda}(x-s))}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k}
\end{aligned}$$

and if $x \in [\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]$ with $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ] - \lim x = s$, then it follows that $x \in [\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]_\theta$ with $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]_\theta - \lim x = s$. $\square$

Theorem 2.3. *Let $\theta = (k_r)$ be a lacunary sequence with $\limsup q_r < \infty$. Then $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]_\theta \subset [\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]$ and $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]_\theta - \lim x = [\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ] - \lim x$.*

Proof. Let $x \in [\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]_\theta$ with $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]_\theta - \lim x = s$. Then for $\epsilon > 0$, there exists j_0 such that for every $j \geq j_0$ and all m,

$$g_{jm} = \frac{1}{h_j} \sum_{i \in I_j} M_k \Big(\| \frac{t_{im}(B^{\mu}_{\Lambda}(x-s))}{\rho}, z_1, \cdots, z_{n-1} \| \Big)^{p_k} < \epsilon,$$

that is, we can find some positive constant C, such that

$$g_{jm} < C \tag{2.1}$$

for all j and m, $\limsup q_r < \infty$ implies that there exists some positive number K such that

$$q_r < K \text{ for all } r \geq 1. \tag{2.2}$$

Therefore, for $k_{r-1} < n \leq k_r$, we have by (2.1) and (2.2)

$$\begin{aligned}
\frac{1}{n+1} & \sum_{i=0}^{n} M_k \Big(\| \frac{t_{im}(B^{\mu}_{\Lambda}(x-s))}{\rho}, z_1, \cdots, z_{n-1} \| \Big)^{p_k} \\
\leq & \frac{1}{k_{r-1}} \sum_{i=0}^{k_r} M_k \Big(\| \frac{t_{im}(B^{\mu}_{\Lambda}(x-s))}{\rho}, z_1, \cdots z_{n-1} \| \Big)^{p_k} \\
= & \frac{1}{k_{r-1}} \sum_{j=0}^{r} \sum_{i \in I_j} M_k \Big(\| \frac{t_{im}(B^{\mu}_{\Lambda}(x-s))}{\rho}, z_1, z_2, \cdots z_{n-1} \| \Big)^{p_k} \\
= & \frac{1}{k_{r-1}} \Big[\sum_{j=0}^{j_0} \sum_{j=j_0+1}^{r} \Big] \sum_{i \in I_j} M_k \Big(\| \frac{t_{im}(B^{\mu}_{\Lambda}(x-s))}{\rho}, z_1, \cdots, z_{n-1} \| \Big)^{p_k} \\
\leq & \frac{1}{k_{r-1}} \Big(\sup_{l \leq p \leq j_0} g_{pm} \Big) k_{j_0} + \epsilon (k_j - k_{j_0}) \frac{1}{k_{r-1}} \\
\leq & C \frac{k_{J_0}}{k_{r-1}} + \epsilon\, K.
\end{aligned}$$

Since $k_{r-1} \to \infty$, we get $x \in [\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]$ with $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ] - \lim x = s$. This completes the proof of the theorem. $\square$

Theorem 2.4. *Let $1 < \liminf q_r \leq \limsup q_r < \infty$. Then $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ] = [\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]_\theta$.*

Proof. It follows directly from Theorem 2.2. and Theorem 2.3. So we omit the details. □

Theorem 2.5. *Let* $x \in [\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ] \cap [\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]_{\theta}$. *Then* $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ] - \lim x = [\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]_{\theta} - \lim x$ *and* $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]_{\theta} - \lim x$ *is unique for any lacunary sequence* $\theta = (k_r)$.

Proof. Let $x \in [\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ] \cap [\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]_{\theta}$. and $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]_{\theta} - \lim x = s$, $[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]_{\theta} - \lim x = s'$. Suppose $s \neq s'$. We see that

$$\begin{aligned} M_k\Big(||\frac{s-s^{'}}{\rho}, z_1, \cdot\cdot, z_n||\Big)^{p_k} &\leq \frac{1}{h_r}\sum_{i\in I_r} M_k\Big(||\frac{t_{im}(B^{\mu}_{\Lambda}(x-s))}{\rho}, z_1, \cdots, z_n||\Big)^{p_k} \\ &+ \frac{1}{h_r}\sum_{i\in I_r} M_k\Big(||\frac{t_{im}(B^{\mu}_{\Lambda}(x-s^{'}))}{\rho}, z_1, \cdots, z_n||\Big)^{p_k} \\ &\leq \lim_r \sup_m \frac{1}{h_r}\sum_{i\in I_r} M_k\Big(||\frac{t_{im}(B^{\mu}_{\Lambda}(x-s))}{\rho}, z_1, \cdots, z_{n-1}\Big)^{p_k} + 0. \end{aligned}$$

Hence there exists r_0, such that for $r > r_0$,

$$\frac{1}{h_r}\sum_{i\in I_r} M_k\Big(||\frac{t_{im}(B^{\mu}_{\Lambda}(x-s))}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} > \frac{1}{2} M_k\Big(||\frac{s-s^{'}}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k}.$$

Since $\Big[\hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p), ||., \cdots, .||\Big] - \lim x = s$, it follows that

$$\begin{aligned} 0 &\geq \limsup \frac{h_r}{k_r} M_k\Big(||\frac{s-s^{'}}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} \\ &\geq \liminf \frac{h_r}{k_r} M_k\Big(||\frac{s-s^{'}}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} \\ &\geq 0 \end{aligned}$$

and so, $\lim q_r = 1$. Hence by Theorem 2.3.,

$$[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]_{\theta} \subset [\ \hat{w}(\mathcal{M}, p, \|., .., .\|)\ ]$$

and

$$[\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ]_{\theta} - \lim x = s^{'} = s = [\ \hat{w}(\mathcal{M}, B^{\mu}_{\Lambda}, p, \|., .., .\|)\ ] - \lim x.$$

Further,

$$\begin{aligned}
& \tfrac{1}{n+1} \sum_{i=0}^{n} M_k\Big(||\frac{t_{im}(B_\Lambda^\mu(x-s))}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} \\
+ & \frac{1}{n+1}\sum_{i=0}^{n} M_k\Big(||\frac{t_{im}(B_\Lambda^\mu(x-s^{'}))}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} \\
\geq & M_k\Big(||\frac{s-s^{'}}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} \\
\geq & 0
\end{aligned}$$

and taking the limit on both sides as $n \to \infty$, we have

$$M_k\Big(||\frac{s-s^{'}}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} = 0$$

that is $s = s^{'}$ for Musielak-Orlicz function $\mathcal{M} = (M_k)$ and this completes the proof of the theorem.

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